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Distinct distances between points and lines ☆

Micha Sharir [a,1], Shakhar Smorodinsky [b,c,*,2], Claudiu Valculescu [c,3], Frank de Zeeuw [c,*,3]

[a] School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel
[b] Department of Mathematics, Ben-Gurion University of the NEGEV, Be'er Sheva 84105, Israel
[c] EPFL, Switzerland

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ABSTRACT

We show that for m points and n lines in $\mathbb{R}^2$, the number of distinct distances between the points and the lines is $\Omega(m^{1/5}n^{3/5})$, as long as $m^{1/2} \le n \le m^2$. We also prove that for any m points in the plane, not all on a line, the number of distances between these points and the lines that they span is $\Omega(m^{4/3})$. The problem of bounding the number of distinct point-line distances can be reduced to the problem of bounding the number of tangent pairs among a finite set of lines and a finite set of circles in the plane, and we believe that this latter question is of independent interest. In the same vein, we show that n circles in the plane determine at most $O(n^{3/2})$ points where two or more circles are tangent, improving the previously best known bound of $O(n^{3/2}\log n)$. Finally, we study three-dimensional versions of the distinct point-line distances problem, namely, distinct point-line distances and distinct point-plane distances. The problems studied in this paper are all new, and the bounds that we derive for them, albeit most likely not tight, are non-trivial to prove. We hope that our work will motivate further studies of these and related problems.

1. Introduction

In 1946 Paul Erdős [7] posed the following two problems, which later became extremely influential and central questions in combinatorial geometry: the so-called *repeated distances* and *distinct distances* problems. The first problem deals with the maximum number of repeated distances in a set of n points in the plane, or in other words, the maximum number of pairs of points at some fixed distance. The best known upper bound is $O(n^{4/3})$ [20], but the best known lower bound, attained by the $\sqrt{n} \times \sqrt{n}$ grid, is only $n^{1+\frac{\Omega(1)}{\log\log n}}$ [7]. The second problem asks for the minimum number of distinct distances determined by a set of n points in the plane. Guth and Katz [10] proved that the number of distinct distances is $\Omega(n/\log n)$ for any set

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* Corresponding authors.
E-mail address: shakhar@math.bgu.ac.il (S. Smorodinsky).
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of n points in the plane. This bound is nearly tight in the worst case, since an upper bound $O(n/\sqrt{\log n})$ is attained by the $\sqrt{n} \times \sqrt{n}$ grid [7].

In this paper we consider questions similar to those above, but for distances *between points and lines*. To be precise, we define the distance between a point p and a line ℓ to be the minimum Euclidean distance between p and a point of ℓ.

Let P be a set of m points and L a set of n lines in the plane. Let $I(P, L)$ denote the number of incidences between P and L, i.e., the number of pairs $(p, \ell) \in P \times L$ such that the point p lies on the line ℓ. The classical result of Szemerédi and Trotter [22] asserts that

$$I(P, L) = O\left(m^{2/3}n^{2/3} + m + n\right). \tag{1}$$

This bound is tight in the worst case, by constructions due to Erdős and to Elekes. See the survey of Pach and Sharir [15] for more details on geometric incidences.

The point-line incidence setup can be viewed as a special instance of a repeated distance problem between points and lines. Specifically, the Szemerédi–Trotter result provides a sharp bound on the number of point-line pairs such that the point is at distance 0 from the line. As a matter of fact, the same bound holds if we consider pairs $(p, \ell) \in P \times L$ that have any fixed positive distance, say 1. Indeed, replace each line $\ell \in L$ by a pair ℓ^+, ℓ^- of lines parallel to ℓ and at distance 1 from it. Then any point $p \in P$ at distance 1 from ℓ must lie on one of these lines. Hence the number of point-line pairs at distance 1 is at most the number of incidences between the m points of P and the $2n$ lines ℓ^+, ℓ^-, for $\ell \in L$. (Actually, a line in the shifted set might arise twice, but this does not affect the asymptotic bound.) This proves that the number of times a single distance can occur between m points and n lines is $O\left(m^{2/3}n^{2/3} + m + n\right)$, and this bound is easily seen to be tight in the worst case. Indeed, as mentioned above, there are sets of points and lines with this number of incidences, and by replacing each line with a parallel line at distance 1, we get a construction with $\Theta\left(m^{2/3}n^{2/3} + m + n\right)$ repeated point-line distances.

Distinct point-line distances. Our first main result concerns distinct distances between m points and n lines in the plane. In contrast with the repeated distances question, as discussed above, the distinct distances variant seems harder than for point-point distances, and the lower bound that we are able to derive is inferior to that of [10]. Nevertheless, deriving this bound is not an easy task, and follows by a combination of several advanced tools from incidence geometry. We hope that our work will trigger further research into this problem.

We write $D(m, n)$ for the minimum number of point-line distances determined by a set of m points and a set of n lines in $\mathbb{R}^2$. Our first main theorem is the following lower bound for $D(m, n)$.

Theorem 1.1. *For $m^{1/2} \le n \le m^2$, the minimum number $D(m, n)$ of point-line distances between m points and n lines in $\mathbb{R}^2$ satisfies*

$$D(m, n) = \Omega\left(m^{1/5}n^{3/5}\right).$$

Our proof also yields a stronger statement: For any set P of m points, and any set L of n lines in the plane, with $m^{1/2} \le n \le m^2$, there always exists a point $p \in P$ such that the number of distinct distances from p to L satisfies the bound in Theorem 1.1.

We note that the upper bound $D(m, n) \le \lceil n/2 \rceil$ is easy to achieve by the following construction. Place n parallel lines, say the horizontal lines $y = j$ for integers $j = 1, \dots, n$. Place all points on the line $y = n/2 + 1/2$. Since all points on the median line have the same distance from any given horizontal line, the number of distinct point-lines distances is $\lceil n/2 \rceil$. Note that this upper bound does indeed dominate the bound in Theorem 1.1, as long as $m^{1/2} \le n$.

As a first step towards Theorem 1.1, we study the problem of bounding from above the number of tangencies between n lines and k circles; see Theorem 1.3 below. This results in a bound that is somewhat weaker than that in Theorem 1.1. To arrive at the stronger bound in Theorem 1.1, we adapt an idea used by Székely [21] to prove the lower bound $\Omega(n^{4/5})$ for the number of distinct point-point distances determined by n points.

The bound of Székely was improved by Solymosi and Tóth [18] and by Guth and Katz [10], but it appears difficult to adapt their proofs to the point-line distances setup. Let us explain the problem in the case of the Guth–Katz proof. A key step in that proof is a transformation from a pair of points p, q to the set of rigid motions that map p to q. In the right parametrization, this set is a line in $\mathbb{R}^3$, and distinct pairs of points correspond to distinct lines. To do the same for point-line distances, we would transform a pair of lines ℓ, ℓ' to the rigid motions that map ℓ to ℓ', which is again a line in $\mathbb{R}^3$. However, it is not true that distinct pairs of lines always yield distinct lines in $\mathbb{R}^3$, and this ruins the Guth–Katz approach. It thus appears that the point-line distances problem is in some sense harder, and obtaining a better bound seems to require significant new ideas.

Distinct distances between points and their spanned lines. We also study the number of distinct point-line distances between a finite set of non-collinear points (that is, not all points lie on a common line) and the set of lines that they span. This question has a different flavor, because the number of lines spanned by m non-collinear points varies from m to $\binom{m}{2}$. When the points span many lines, Theorem 1.1 provides a reasonable bound, which can be as large as $\Omega(m^{7/5})$, but when the points span few lines, the resulting bound is a relatively weak $\Omega(m^{4/5})$. We use a different approach to obtain a better overall bound.