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T-shape visibility representations of 1-planar graphs

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ABSTRACT

A shape visibility representation displays a graph so that each vertex is represented by an orthogonal polygon of a particular shape and for each edge there is a horizontal or vertical line of sight of width $\epsilon > 0$ between the polygons assigned to its endvertices. Special shapes are rectangles, L, T, E, and H-shapes, and caterpillars. A graph is 1-planar if there is a drawing in the plane such that each edge is crossed at most once and is IC-planar if in addition no two crossed edges share a vertex.

We show that every IC-planar graph has a flat rectangle visibility representation, in which each rectangle has height ϵ, and that every 1-planar graph has a T-shape visibility representation. The representations use quadratic area and can be computed in linear time from a given embedding.

1. Introduction

A graph is commonly visualized by a drawing in the plane or on another surface. In return, properties of drawings are used to define properties of graphs. Planar graphs are the most prominent example. Also, the genus of a graph and k-planar graphs are defined in this way, where a graph is k-planar for some $k \geq 0$ if there is a drawing in the plane such that each edge is crossed at most k times.

1-planar graphs are the most important class of so-called beyond-planar graphs, see [37]. Beyond-planarity comprises graph classes that extend the planar graphs and are defined by specific restrictions on crossings. 1-planar graphs were studied first by Ringel [42] who showed that they are at most 7-colorable. In fact, 1-planar graphs are 6-colorable [11]. Bodendiek et al. [9,10] observed that 1-planar graphs of size n have at most $4n-8$ edges and that this bound is tight for $n = 8$ and all $n \geq 10$. This fact was discovered independently in many works. In consequence, an embedding has linear size and can be treated in linear time.

IC-planar (independent crossing planar) graphs are an important special case [2]. A graph is *IC-planar* if it admits a drawing so that each edge is crossed at most once and each vertex is incident to at most one crossing edge. IC-planar graphs have at most $3.25n-6$ edges [39] and the bound is tight. In between are NIC-planar graphs [50], which are defined by 1-planar drawings in which two pairs of crossing edges share at most one vertex. They have at most $3.6(n-2)$ edges. 1-planar, NIC-planar, and IC-planar graphs have some properties in common: First, there is a difference between densest and sparsest graphs. A graph is *densest* (*sparsest*) if it cannot be augmented by another edge without violating the defining properties of a graph class and has as many (few) edges as possible. It is known that there are sparsest 1-planar graphs with $\frac{45}{17}n - \frac{84}{17}$ edges [16], sparsest NIC-planar graphs with $3.2(n-2)$ [5] edges and sparsest IC-planar graphs with $3n-4$ edges [5]. The NP-hardness of the recognition problems was discovered independently multiple times [31,38,4,5,15] and

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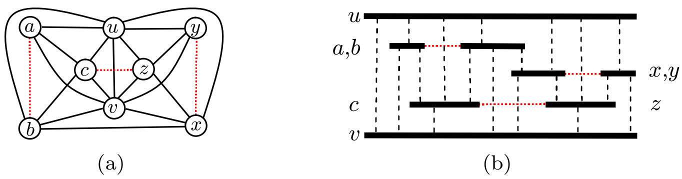

Fig. 1. (a) A 1-planar graph and (b) a (flat) bar visibility representation with vertical and horizontal lines of sight that are drawn as black and red and dotted lines, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

holds even if the graphs are 3-connected and are given with a rotation system. On the other hand, triangulated graphs can be recognized in cubic time [18,13]. A *triangulated graph* admits a drawing so that all faces are triangles. Then all pairs of crossing edges induce K_4 as a subgraph.

The most remarkable distinction between IC-planar and NIC-planar graphs is their relationship to RAC graphs. A graph is RAC (right angle crossing) [24] if it admits a straight-line drawing such that edges cross at a right angle. RAC graphs have at most $4n-10$ edges, and if they meet the upper bound, then they are 1-planar [26]. In contrast, there are 1-planar graphs that are not RAC and RAC graphs that are not 1-planar [24]. Hence, 1-planar graphs and RAC graphs are incomparable. Recently, Brandenburg et al. [15] showed that every IC-planar graph is a RAC graph and Bachmaier et al. [5] proved that RAC graphs and NIC-planar graphs are incomparable. We introduce a class of graphs that is 1-planar and RAC.

Planar graphs admit a visualization by bar visibility representations, as shown in Fig. 8. A *bar visibility representation* consists of a set of non-intersecting horizontal line segments, called bars, and vertical lines of sight between the bars. We assume ϵ-visibility with lines of sight have width $\epsilon > 0$ [21]. Each bar represents a vertex of a graph and there is an edge if (or if and only if) there is a line of sight between the bars of the endvertices. Hence, there is a bijection between vertices and bars and a correspondence between edges and lines of sight that is one-to-one in the *weak* or "if"-version and also onto in the *strong* or "if and only if"-version. A graph is a *bar visibility graph* if it admits a bar visibility representation.

Bar visibility representations and graphs were intensively studied in the 1980s and the representations of planar graphs were discovered independently multiple times [25,41,43,47,49]. Note that strong visibility with lines of sight of width zero excludes $K_{2,3}$ and some 3-connected planar graphs [3] and implies an NP-hard recognition problem [3]. Obviously, every weak visibility graph is an induced subgraph of a strong visibility graph with lines of sight of width zero or $\epsilon > 0$.

In the late 1990s visibility representations were generalized to represent non-planar graphs. The approach by Dean et al. [20] admits semi-transparent bars and lines of sight that traverse up to k other bars. In other words, an edge can cross up to k vertices. Some facts are known about bar k-visibility graphs: for $k=1$ each graph of size n has at most $6n-20$ edges and the bound can be achieved for all $n \geq 8$ [20]. In consequence, K_8 is the largest complete bar 1-visibility graph. A graph has *thickness* k if it can be decomposed into k planar graphs. However, bar 1-visibility graphs are incomparable to thickness two (or biplanar) graphs, since $K_9 - e$ has thickness two and is not a bar 1-visibility graph [34] and conversely there are bar 1-visibility graphs with thickness three [30]. Bar 1-visibility graphs have an NP-hard recognition problem [17]. Last but not least, every 1-planar graph has a bar 1-visibility representation that uses only quadratic area and can be specialized so that a line of sight crosses at most one bar and each bar is crossed at most once [12]. The inclusion relation between 1-planar and bar 1-visibility graphs was obtained independently by Evans et al. [27].

Rectangle visibility representations of graphs were introduced by Hutchinson et al. [34]. Here each vertex is represented by an axis-aligned rectangle and there are horizontal and vertical lines of sight for the edges, which cannot penetrate rectangles. Two rectangles do not intersect. Hutchinson et al. studied the strong version of visibility. They proved a density of $6n-20$, which is tight for all $n \geq 8$. In consequence, K_8 is the largest rectangle visibility graph. Rectangle visibility graphs have thickness two whereas it is unknown whether they have geometric thickness two [34], which requires a decomposition into two straight-line planar graphs. The recognition problem for weak rectangle visibility graphs is NP-hard [45].

We generalize rectangle visibility representations to *σ-shape visibility representations*. A *shape* σ is an orthogonal drawing of a ternary tree τ, which is expanded to an *orthogonal polygon* in a σ-shape visibility representation such that two polygons do not intersect. Thereby, each edge of τ is expanded to a rectangle of width $w > 0$ and height $h > 0$. The images of the vertices are similar and differ only in the length and width of the horizontal and vertical pieces of the polygon. In particular, rectangle visibility is "I"-shape or "–"-shape visibility. Since visibility representations can be reflected or rotated by multiples of 90 degrees, we treat the respective shapes as equivalent and shall identify them. For example, any single element of the set $\{\lfloor, \rfloor, \lceil, \rceil\}$ can be used for an L-shape. However, a set of shapes must be used if the vertices have different shapes, e.g., $\{\lfloor, \rfloor, \lceil, \rceil\}$ for L-shapes in [40]. Other common shapes are H, F or E. A *rake* is a generalized E with a vertical and many horizontal lines to one side of the vertical line. A *caterpillar* is a two-sided rake with horizontal lines to both sides. The number of horizontal lines is reflected by the vertex complexity of ortho-polygon visibility representations [23].

In a *flat rectangle visibility representation*, the rectangles have height ϵ, where ϵ is the width of a line of sight [47], as shown in Fig. 1(b). Then the vertices are represented by bars of height ϵ, as in bar visibility representations, such that two bars at the same level can see one another by a horizontal line of sight if there is no third bar in between. Moreover, a horizontal and a vertical line of sight may cross, which is not allowed in the flat visibility representations by Biedl [7].

Shape visibility representations have been introduced by Di Giacomo et al. [23]. They use caterpillars as shapes in their results. L-visibility representations have been introduced by Evans et al. [28]. Their approach was adopted by Liotta and Montecchiani [40] for the representation of IC-planar graphs.