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On triangle cover contact graphs

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ABSTRACT

Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of pairwise disjoint geometric objects of some type in a $2D$ plane and let $C = \{c_1, c_2, \ldots, c_n\}$ be a set of closed objects of some type in the same plane with the property that each element in C covers exactly one element in S and any two elements in C are interior-disjoint. We call an element in S a *seed* and an element in C a *cover*. A *cover contact graph (CCG)* has a vertex for each element of C and an edge between two vertices whenever the corresponding cover elements touch. It is known how to construct, for any given point seed set, a disk or triangle cover whose contact graph is 1- or 2-connected but the problem of deciding whether a k-connected CCG can be constructed or not for $k > 2$ is still unsolved. A *triangle cover contact graph (TCCG)* is a cover contact graph whose cover elements are triangles. In this paper, we give algorithms to construct a 3-connected $TCCG$ and a 4-connected $TCCG$ for a given set of point seeds. We also show that any connected outerplanar graph has a realization as a $TCCG$ on a given set of collinear point seeds. Note that, under this restriction, only trees and cycles are known to be realizable as CCG.

1. Introduction

Let $S = \{p_1, p_2, \ldots, p_n\}$ be a set of pairwise disjoint geometric objects of some type in the plane and let $C = \{c_1, c_2, \ldots, c_n\}$ be a set of closed objects of some type in the same plane with the property that each element in C covers exactly one element in S and any two elements in C can intersect only on their boundaries. We call an element in S a *seed* and an element in C a *cover*. The seeds may be points, disks or triangles and covering elements may be disks or triangles. The *cover contact graph (CCG)* consists of a set of vertices and a set of edges where each vertex corresponds to a cover and each edge corresponds to a connection between two covers if they touch at their boundaries. In other words, two vertices of a cover contact graph are adjacent if the corresponding cover elements touch at their boundaries. Note that the vertices of the cover contact graph are in one-to-one correspondence to both seeds and covering objects. In a cover contact graph, if disks are used as covers then it is called a *disk cover contact graph* and if triangles are used as covers then it is called a *triangle cover contact graph (TCCG)*. Fig. 1(b) depicts the disk cover contact graph induced by the disk covers in Fig. 1(a),

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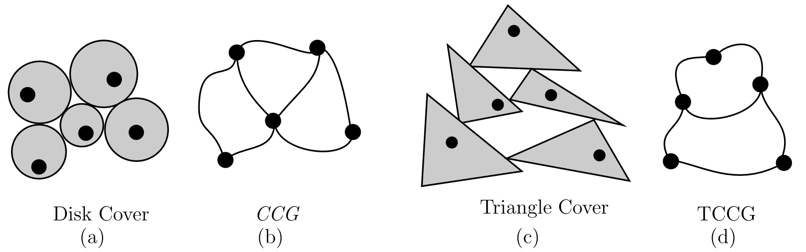

Fig. 1. Illustration for CCG and $TCCG$; (a) a disk cover, (b) a CCG, (c) a triangle cover and (d) a TCCG.

whereas Fig. 1(d) depicts the triangle cover contact graph induced by the triangle covers in Fig. 1(c). A *coin graph* is a graph formed by a set of disks, no two of which have overlapping interiors, by making a vertex for each circle and an edge for each pair of circles that touches. Koebe's theorem [7,9] states that every planar graph can be represented as a coin graph. There are several works [10,11,5] in the geometric-optimization community where the problem is how to cover geometric objects such as points by other geometric objects such as convex shapes, disks. The main goal is to minimize the radius of a set of k disks to cover n input points. Applications of such covering problems are found in geometric optimization problems such as facility location problems [10,11]. Abellanas et al. [1] worked on a "coin placement problem," which is NP-complete. They tried to cover n points using n disks (each having different radius) by placing each disk in the center position at one of the points so that no two disks overlap. Further Abellanas et al. [2] considered another related problem. They showed that for a given set of points in the plane, it is also NP-complete to decide whether there are disjoint disks centered at the points such that the contact graph of the disks is connected.

Recently, Atienza et al. [3] introduced the concept of cover contact graphs where the seeds are not necessarily the center of the disks. They gave an $O(n \log n)$ time algorithm to decide whether a given set of point seeds can be covered with homothetic triangles or disks such that the resulting cover contact graph is 1- or 2-connected. The k-connectivity problem is still unsolved for $k > 2$. Atienza et al. [3] also considered the problem from another direction which they called "realization problem." In a realization problem we are given a graph G of n vertices and a set S of n seeds and we are asked whether there is any covering so that the resulting cover contact graph is G. They gave some necessary conditions and then showed that it is NP-hard to decide whether a given graph can be realized as a disk cover contact graph if the correspondence between vertices and point seeds is given. They also showed that every tree and cycle have realizations as CCGs on a given set of collinear point seeds. Durocher et al. [5] considered a circular cover contact graph problem defined by Atienza et al. [3]. They showed that when the input discs and the covering discs are all constrained to touch a line, then the problem of deciding whether the input set has a connected CCG is NP-hard. They also defined an approximate variation of the problem, where the covering discs are allowed to overlap by a small amount. They gave a polynomial-time algorithm such that if there exists an exact solution to the problem, then the algorithm returns an ϵ-approximate solution.

In this paper, we consider a set of arbitrary seeds in the plane where the seeds are points and the covers are triangles. First we consider the set of seeds which are in general position, i.e., no two seeds lie on a vertical line and we give an $O(n \log n)$ algorithm to construct a 3-connected $TCCG$ of the set of seeds. We also give a $O(n \log n)$ algorithm to construct a 4-connected $TCCG$ for a given set of six or more seeds. Addressing the realization problem, we give an algorithm that realizes a given outerplanar graph as a triangle cover contact graph ($TCCG$) for a given set of seeds on a line.

The remaining of the paper is organized as follows. Section 2 presents some definitions and preliminary results. Section 3 gives algorithms to construct a 3-connected $TCCG$ and 4-connected $TCCG$. Section 4 gives an algorithm that realizes a given outerplanar graph as $TCCG$. Finally, Section 5 concludes the paper by suggesting some future works. A preliminary version of this paper was presented at WALCOM 2015 [6].

2. Preliminaries

In this section we present some terminologies and definitions which will be used throughout the paper. For the graph theoretic definitions which have not been described here, see [4,8].

A graph is *planar* if it can be embedded in the plane without edge crossing except at the vertices where the edges are incident. A *plane graph* is a planar graph with a fixed planar embedding. A plane graph divides the plane into connected regions called *faces*. The unbounded region is called the *outer face*; the other faces are called *inner faces*. The cycle lies on the outer face is called *outer cycle*. A plane graph G is an *outerplanar graph* if all vertices of G lie on the outer face.

The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected graph or a single-vertex graph. We say that G is k-connected if $\kappa(G) \geq k$. A vertex v in a connected graph G is a *cut-vertex* if the deletion of v from G results in a disconnected graph. Similarly an edge e in a connected graph G is a *bridge* if the deletion of e from G results in a disconnected graph. A 2-connected or biconnected graph does not contain any cut vertex.

A *biconnected component* of a connected graph G is a maximal biconnected subgraph of G. A *block* of a connected graph G is either a biconnected component or a bridge of G. The graph in Fig. 2(a) has the blocks $B_0, B_1, \ldots, B_8$ depicted in Fig. 2(b). The blocks and cut vertices in G can be represented by a tree T, called the *BC-tree* of G. In T each block is represented by