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Combinatorics and complexity of guarding polygons with edge and point 2-transmitters [☆]

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A R T I C L E   I N F O

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In memory of Ferran, who effused thoughtfulness and generosity, and to whom we owe an abundance of gratitude for his many wonderful contributions to the field of computational geometry

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A B S T R A C T

We consider a generalization of the classical Art Gallery Problem, where instead of a light source, the guards, called *k-transmitters*, model a wireless device with a signal that can pass through at most k walls. We show it is NP-hard to compute a minimum cover of point 2-transmitters, point k-transmitters, and edge 2-transmitters in a simple polygon. The point 2-transmitter result extends to orthogonal polygons. In addition, we give necessity and sufficiency results for the number of edge 2-transmitters in general, monotone, orthogonal monotone, and orthogonal polygons.

1. Introduction

The traditional art gallery problem (AGP) considers placing guards in an art gallery—modeled by a polygon—so that every point in the room can be seen by some guard. A similar question asks how to place wireless routers so that an entire room has a strong signal. Observation shows that often not only the distance from a modem, but also the number of walls a signal has to pass through, influences signal strength.

☆ Abstracts of part of this work appeared in the informal workshops FWCG [1] and EuroCG [2].

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Aichholzer et al. [3] first formalized this problem by considering *k-modems* (*k-transmitters*), devices whose wireless signal can pass through at most k walls. Since 2010, little progress has been made on the problem of k-transmitters, or even the problem of 2-transmitters, despite reaching a wide audience as the topic of a computational geometry column by Joseph O'Rourke [4] in the SIGACT News in 2012. Analogous to the original AGP ($k = 0$), two main questions can be considered:

(1) Given a polygon P, can a minimum cardinality k-transmitter cover be computed efficiently?
(2) Given a class of polygons of n vertices, what are lower and upper bounds on the number of guards needed to cover a polygon from this class?

For the classical AGP, the complexity question (1) was answered with NP-hardness for many variants. O'Rourke and Supowit [5] gave a reduction from 3SAT, for polygons with holes and guards restricted to lie on vertices. Lee and Lin [6] gave the result for simple polygons. This result was extended to point guards (that are allowed to be located anywhere inside of P) by Aggarwal (see [7]); Schuchardt and Hecker [8] gave NP-hardness proofs for rectilinear simple polygons, both for point and vertex guards. The complexity of the k-/2-transmitter problem had not previously been settled, and in this paper, we prove the minimum point 2-transmitter, the minimum point k-transmitter, and the minimum edge 2-transmitter problems to be NP-hard in simple polygons. The minimum point 2-transmitter result also holds for simple, orthogonal polygons.

Answers to (2) are often referred to as "Art Gallery theorems", e.g. Chvátal's tight bound of $\lfloor \frac{n}{3} \rfloor$ for simple polygons [9]. Fisk [10] later gave a short and simple proof for Chvátal's result. In the case of orthogonal polygons, the bound becomes $\lfloor \frac{n}{4} \rfloor$, as shown by Kahn et al. [11].

For k-transmitters, Aichholzer et al. [3] showed $\lceil \frac{n}{2k} \rceil$ k-transmitters are always sufficient and $\lceil \frac{n}{2k+4} \rceil$ k-transmitters are sometimes necessary to cover a monotone n-gon[4]; for monotone orthogonal polygons they gave a tight bound of $\lceil \frac{n-2}{2k+4} \rceil$ k-transmitters, for k even and $k = 1$. Fabila-Monroy et al. [12] improved the bounds on monotone polygons to a tight value of $\lceil \frac{n-2}{2k+3} \rceil$. In addition, they gave tight bounds for monotone orthogonal polygons for all values of k. Other publications explored k-transmitter coverage of regions other than simple polygons, such as coverage of the plane in the presence of line or line segment obstacles [13,14]. For example, Ballinger et al. [13] established that for disjoint segments in the plane, where each segment has one of two slopes and the entire plane is to be covered, $\lceil \frac{1}{2} (\frac{5}{6}^{\log(k+1)} n + 1) \rceil$ k-transmitters are always sufficient, and $\lceil \frac{n+1}{2k+2} \rceil$ k-transmitters are sometimes necessary. For polygons, Ballinger et al. concentrated on a class of spiral polygons, so called *spirangles*, and established that $\lfloor \frac{n}{8} \rfloor$ 2-transmitters are necessary and sufficient. For simple n-gons the authors provided a lower bound of $\lfloor n/6 \rfloor$ 2-transmitters. We improve this bound in Section 4.

For the classical AGP variants involving guards with different capabilities have been considered; for example, *edge guards* monitor each point of the polygon that is visible to some point of the edge. The computational complexity of the minimum edge guard problem was settled by Lee and Lin [6] who proved it to be NP-hard. Bjorling-Sachs [15] showed a tight bound of $\lfloor \frac{3n+4}{16} \rfloor$ edge guards for orthogonal polygons. For general polygons $\lfloor \frac{3n}{10} \rfloor + 1$ edge guards are always sufficient and $\lfloor \frac{n}{4} \rfloor$ are sometimes necessary [16], and no tighter bounds are known.

Other problems related to k-transmitter coverage have also been considered. Already in 1988, Dean et al. [17] considered a problem in which single edges become transparent. While for ordinary visibility the AGP equates to finding a cover of star-shaped polygons, Dean et al. defined pseudo-star-shaped polygons to include parts that are visible through single edges. The authors concentrated on testing whether a polygon is pseudo-star-shaped, that is, whether there exists one of these more powerful guards that completely covers the input polygon. Moreover, Mouawad and Shermer [18] considered the so-called *Superman problem*: given a polygon P and its subpolygon K, for a point x in the exterior of P, how many edges of P must be made intransparent or opaque so that x cannot see a point of K?

Our Results. Our focus is on finding covers of lower power transmitters, that is, mainly 2-transmitters. This is in line with the work of Ballinger et al. [13] and is motivated both by practical applications and by virtue of being the natural extension of classical Art Gallery results, that is, results for $k = 0$. Abstracts of part of this work appeared in two informal workshops ([1,2]).

We provide NP-hardness results for several problem variants in simple polygons in Section 3. In Section 4 we provide observations on point 2-transmitter covers and a lower bound for the number of point 2-transmitters in general polygons. We give sufficiency and necessity results for edge 2-transmitters in Section 5; these results are summarized in Table 1.

2. Notations and preliminaries

In a polygon P, a point $q \in P$ is *2-visible* from $p \in \mathbb{R}^2$ if the straight-line segment $\overline{pq}$ intersects P in at most two connected components.

For a point $p \in P$, we define the *2-visibility region* of p, $2\mathrm{VR}(p)$, as the set of points in P that are 2-visible from p. For a set $S \subseteq P$, $2\mathrm{VR}(S) := \cup_{p \in S} 2\mathrm{VR}(p)$. A set $C \subseteq P$ is a *2-transmitter cover* if $2\mathrm{VR}(C) = P$.

[4] The stated lower bound of $\lceil n/(2k+2) \rceil$ given in [3] is a typo, and the example only necessitates $\lceil n/(2k+4) \rceil$ 2-transmitters.