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Forest-like abstract Voronoi diagrams in linear time ☆

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This paper is dedicated to Ferran Hurtado

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ABSTRACT

Abstract Voronoi diagrams are a general framework covering many types of concrete diagrams for different types of sites or distance measures. Generalizing a famous result by Aggarwal et al. [1] we prove the following. Suppose it is known that inside a closed domain D the Voronoi diagram $V(S)$ is a tree, and for each subset $S' \subset S$, a forest with one face per site. If the order of Voronoi regions of $V(S)$ along the boundary of D is given, then $V(S)$ inside D can be constructed in linear time.

1. Introduction

Voronoi diagrams [2,6] are well-studied structures providing proximity information used in many different engineering and science applications [12]. In general, $O(n \log n)$ time is necessary and sufficient to construct a Voronoi diagram of n sites.

In 1987, Aggarwal et al. [1] presented a paper that solved an outstanding open problem. They showed how to compute, in linear time, the Voronoi diagram of the vertices of a convex polygon. Their technique implied an optimal solution to another difficult problem, namely how to delete a point site from a Voronoi diagram in time linear in the number of its Voronoi neighbors (which need not be in convex position).

Considering distance measures different from the Euclidean metric, or sites different from points, a natural question is the following. *For which other types of Voronoi diagrams can the deletion of a site also be implemented in linear time?*

We believe that abstract Voronoi diagrams [8,9] are the appropriate framework to study this question, because their definition does not depend on the notions of sites and distance (which vary anyway) but on bisecting curves and their combinatorial properties. In this paper S denotes a set of n symbolic sites. For each pair of sites p and q from S one takes an unbounded curve $J(p,q) = J(q,p)$ as primary object, together with the open domains $D(p,q)$ and $D(q,p)$ it separates. Abstract Voronoi regions are defined by

$$\mathrm{VR}(p,S) := \bigcap_{q \in S \setminus \{p\}} D(p,q)$$

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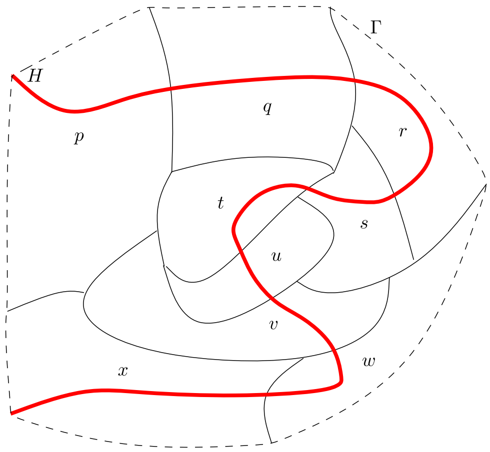

Fig. 1. A Hamiltonian abstract Voronoi diagram, with respect to H. The sites are ordered $p, q, r, s, t, u, v, w, x$ along the Hamiltonian path H.

and the abstract Voronoi diagram by

$$V(S) := \mathbb{R}^2 \setminus \bigcup_{p \in S} \mathrm{VR}(p, S).$$

The following axioms are required to hold for each subset S' of S of size 3.

(A1) *Each curve $J(p, q)$, where $p \neq q$, is unbounded. After stereographic projection to the sphere, it can be completed to a closed Jordan curve through the north pole.*
(A2) *Each nearest Voronoi region $VR(p, S')$ is nonempty and pathwise connected.*
(A3) *Each point of the plane belongs to the closure of a Voronoi region $VR(p, S')$.*

These axioms imply many properties known for concrete Voronoi diagrams. For example, if two bisecting curves $J(p, q)$ and $J(q, r)$ cross at some point v then $J(p, r)$ must pass through v, too. Voronoi regions are simply-connected, the Voronoi diagram is a plane graph of linear complexity, and it can be constructed in expected time $O(n \log n)$ [11,9].

Now the above question translates into the following. *Is it possible to remove a site from an abstract Voronoi diagram, in time linear in the number of its Voronoi neighbors?* Despite serious effort, this general question is still open. In order to attain linear time, we need to make additional assumptions on the system of bisecting curves. In this paper we report on recent progress in relaxing these assumptions.

Motivated by this question, in [10] the second and third author obtained a linear time algorithm for constructing a "Hamiltonian" type of abstract Voronoi diagram where a curve is given that visits, for each subset S' of S, each Voronoi region in $V(S')$ exactly once. This algorithm used the coloring and selection schemes of the divide&conquer technique in [1]. In order to set the background for the new result presented here, we review the construction from [10] in Section 2.

Next, in Section 3, we show that the problem of determining for a given abstract Voronoi diagram the existence of an unbounded simple curve visiting each Voronoi region exactly once is NP-complete.

Then, in Section 4 we study the following situation. We are given a domain D and want to compute an abstract Voronoi diagram $V(S)$ inside D. We know the order in which the Voronoi regions of $V(S)$ intersect the boundary of D, and we assume that each region contributes exactly one segment to ∂D (which implies that $V(S) \cap D$ is a tree). But as we change to a subset S' of S, some regions may "break through to the other side" of D, due to a lack of opposition. We assume that each Voronoi region of $V(S')$ still has a connected intersection with D (which implies that $V(S') \cap D$ is a forest). These assumptions are fulfilled, for example, for nearest-neighbor Voronoi regions of disjoint line segments, or convex polygons, a situation recently investigated by E. Khramtcova and E. Papadopoulou [7]. Note that farthest-neighbor Voronoi regions of disjoint line segments could be disconnected, while the nearest-neighbor regions considered here are star-shaped, hence connected.

We prove that $V(S) \cap D$ can be computed in linear time if the above assumptions are fulfilled. Our algorithm uses a coloring and selection scheme quite different from the one in [1]. Preliminary versions of this result have appeared in [3] and [4].

2. Hamiltonian abstract Voronoi diagrams

Let H denote a simple, unbounded curve, homeomorphic to a line, that passes exactly once through each region of the AVD $V(S)$. Necessarily, the first and the last region visited by H must be unbounded, compare with Fig. 1. To make it easier to handle unbounded regions and edges, we assume that a large simply closed curve Γ is given around the diagram,

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