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The dual diameter of triangulations ☆

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ABSTRACT

Let $\mathcal{P}$ be a simple polygon with n vertices. The *dual graph* T^* of a triangulation T of $\mathcal{P}$ is the graph whose vertices correspond to the bounded faces of T and whose edges connect those faces of T that share an edge. We consider triangulations of $\mathcal{P}$ that minimize or maximize the diameter of their dual graph. We show that both triangulations can be constructed in $O(n^3 \log n)$ time using dynamic programming. If $\mathcal{P}$ is convex, we show that any minimizing triangulation has dual diameter exactly $2 \cdot \lceil \log_2(n/3) \rceil$ or $2 \cdot \lceil \log_2(n/3) \rceil - 1$, depending on n. Trivially, in this case any maximizing triangulation has dual diameter $n-2$. Furthermore, we investigate the relationship between the dual diameter and the number of *ears* (triangles with exactly two edges incident to the boundary of $\mathcal{P}$) in a triangulation. For convex $\mathcal{P}$, we show that there is always a triangulation that simultaneously minimizes the dual diameter and maximizes the number of ears. In contrast, we give examples of general simple polygons where every triangulation that maximizes the number of ears has dual diameter that is quadratic in the minimum possible value. We also consider the case of point sets in general position in the plane. We show that for any such set of n points there are triangulations with dual diameter in $O(\log n)$ and in $\Omega(\sqrt{n})$.

In memoriam: Ferran Hurtado (1951–2014)

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1. Introduction

Let $\mathcal{P}$ be a simple polygon with $n > 3$ vertices. We regard $\mathcal{P}$ as a closed two-dimensional subset of the plane, containing its boundary. A *triangulation* T of $\mathcal{P}$ is a maximal crossing-free geometric (i.e., straight-line) graph whose vertices are the vertices of $\mathcal{P}$ and whose edges lie inside $\mathcal{P}$. Hence, T is an outerplanar graph. Similarly, for a set S of n points in the plane, a *triangulation* T of S is a maximal crossing-free geometric graph whose vertices are exactly the points of S. It is well known that in both cases all bounded faces of T are triangles. The *dual graph* T^* of T is the graph with a vertex for each bounded face of T and an edge between two vertices if and only if the corresponding triangles share an edge in T. If all vertices of T are incident to the unbounded face, then T^* is a tree. An *ear* in a triangulation of a simple polygon is a triangle whose vertex in the dual graph is a leaf (equivalently, two out of its three edges are edges of $\mathcal{P}$). We call the diameter of the dual graph T^* the *dual diameter (of the triangulation T)*. In the following, we will study combinatorial and algorithmic properties of *minimum* and *maximum dual diameter triangulations* for simple polygons and for planar point sets (minDTs and maxDTs for short). Note that both triangulations need not to be unique.

Previous work Shermer [11] considers *thin* and *bushy* triangulations of simple polygons, i.e., triangulations that minimize or maximize the number of ears. He presents algorithms for computing a thin triangulation in time $O(n^3)$ and a bushy triangulation in time $O(n)$. Shermer also claims that bushy triangulations are useful for finding paths in the dual graph, as is needed, e.g., in geodesic algorithms. In that setting, however, the running time is not actually determined by the number of ears, but by the dual diameter of the triangulation. Thus, bushy triangulations are only useful for geodesic problems if there is a connection between maximizing the number of ears and minimizing the dual diameter. While this holds for convex polygons, we show that, in general, there exist polygons for which no minDT maximizes the number of ears. Moreover, we give examples where forcing a single ear into a triangulation may almost double the dual diameter, and the dual diameter of any bushy triangulation may be quadratic in the dual diameter of a minDT.

The dual diameter also plays a role in the study of edge flips: given a triangulation T, an *edge flip* is the operation of replacing a single edge of T with another one so that the resulting graph is again a valid triangulation. In the case of convex polygons, edge flips correspond to rotations in the dual binary tree [12]. For this case, Hurtado, Noy, and Urrutia [4,13] show that a triangulation with dual diameter k can be transformed into a fan triangulation by a sequence of most k parallel flips (i.e., two edges not incident to a common triangle may be flipped simultaneously). They also obtain a triangulation with logarithmic dual diameter by recursively cutting off a linear number of ears.

While we focus on the dual graph of a triangulation, distance problems in the primal graph have also been considered. For example, Kozma [8] addresses the problem of finding a triangulation that minimizes the total link distance over all vertex pairs. For simple polygons, he gives a sophisticated $O(n^{11})$ time dynamic programming algorithm. Moreover, he shows that the problem is strongly NP-complete for general point sets when arbitrary edge weights are allowed and the length of a path is measured as the sum of the weights of its edges.

Our results In Section 2, we present several properties of the dual diameter for triangulations of simple polygons. Among other results, we calculate the exact dual diameter of minDTs and maxDTs of convex polygons, which can be obtained by maximizing and minimizing the number of ears of the triangulation, respectively. On the other hand, we show that there exist simple polygons where the dual diameter of any minDT is $O(\sqrt{n})$, while that of any triangulation that maximizes the number of ears is in $\Omega(n)$. Likewise, there exist simple polygons where the dual diameter of any triangulation that minimizes the number of ears is in $O(\sqrt{n})$, while the maximum dual diameter is linear. In Section 3, we present efficient algorithms to construct a minDT and a maxDT for any given simple polygon.

Finally, in Section 4 we consider the case of planar point sets, showing that for any point set in the plane in general position there are triangulations with dual diameter in $O(\log n)$ and in $\Omega(\sqrt{n})$, respectively.

2. The number of ears and the diameter

The dual graph of any triangulation T has maximum degree 3. In this case, the so-called *Moore bound* implies that the dual diameter of T is at least $\log_2(\frac{t+2}{3})$, where t is the number of triangles in T (see, e.g., [9]). For convex polygons, we can compute the minimum dual diameter exactly.

Proposition 2.1. *Let $\mathcal{P}$ be a convex polygon with $n \geq 3$ vertices, and let $m \geq 1$ such that $n \in \{3 \cdot 2^{m-1} + 1, \ldots, 3 \cdot 2^m\}$. Then any* minDT *of $\mathcal{P}$ has dual diameter $2 \cdot \lceil \log_2(n/3) \rceil - 1$ if $n \in \{3 \cdot 2^{m-1} + 1, \ldots, 4 \cdot 2^{m-1}\}$, and $2 \cdot \lceil \log_2(n/3) \rceil$ if $n \in \{4 \cdot 2^{m-1} + 1, \ldots, 3 \cdot 2^m\}$, for some $m \geq 1$.*

Proof. The dual graph of any triangulation of $\mathcal{P}$ is a tree with $n - 2$ vertices and maximum degree 3; see Fig. 1(a) for an example. Furthermore, every tree with $n - 2$ vertices and maximum degree 3 is dual to some triangulation of $\mathcal{P}$.

For the upper bound, suppose first that $n = 3 \cdot 2^m$, for some $m \geq 1$. We define a triangulation T_1 as follows. It has a central triangle that splits $\mathcal{P}$ into three sub-polygons, each with 2^m edges on the boundary. For each sub-polygon, the dual tree for T_1 is a full binary tree of height $m - 1$ with 2^{m-1} leaves; see Fig. 1(b). The leaves of T_1^* correspond to the ears of T_1. The shortest path between any two ears in two different sub-polygons has length exactly $2(m - 1) + 2 = 2\log_2(n/3)$.