ARTICLE IN PRESS COMGEO:1487

Computational Geometry ••• (••••) •••–•••

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Minimum rectilinear Steiner tree of n points in the unit square

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ARTICLE INFO

Article history:
Received 27 March 2015
Accepted 24 October 2016
Available online xxxx

Dedicated to the memory of Ferran Hurtado: remembering his kindness and graciousness

Keywords:
Minimum rectilinear Steiner tree
Integer partition
Packing
Covering

ABSTRACT

Chung and Graham conjectured (in 1981) that n points in the unit square $[0,1]^2$ can be connected by a rectilinear Steiner tree of length at most $\sqrt{n}+1$. Here we confirm this conjecture for small values of n, and for some new infinite sequences of values of n (but *not* for all n). As an interesting byproduct we obtain close rational approximations of $\sqrt{n}$ from below, for those n.

1. Introduction

Let S be a finite set of points in the plane. A *Euclidean Steiner tree* (EST) for S is a planar straight line graph spanning S. The *Euclidean Steiner tree problem* asks for the shortest such graph, where the distance between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. The solutions take form of a tree, that includes all the points in S, called *terminals*, or *sites*, along with possibly some extra vertices, called *Steiner points*. In an optimal solution each Steiner point has degree 3, and any two consecutive incident edges form an 120° angle [10]. Obviously, a Euclidean minimum spanning tree (EMST) for a point set can always serve as a possibly suboptimal Euclidean Steiner tree for the same set.

The *rectilinear Steiner tree problem* asks for the shortest Steiner tree where the distance between two points $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ is $|x_1 - x_2| + |y_1 - y_2|$. The solution can be drawn as a *rectilinear Steiner tree* (RST), composed solely of horizontal and vertical edges. The RST problem was first suggested by Hanan [11], who also proved the following structural result regarding optimal solutions. Let $G(S)$ be the grid induced by the point set S by drawing a horizontal and a vertical line through each point of S and retaining only the finite segments between intersection points of these lines (in the axis-aligned bounding box of S). Then there exists a shortest RST for S which uses only segments in $G(S)$ [11]; see also [16, 17].

The following questions were raised by Few [5] in 1955: *What is the greatest length $s(n)$ of a minimum Steiner tree, and what is the greatest length $s_{\llcorner}(n)$ of a minimum rectilinear Steiner tree, of n points in the unit square* $[0,1]^2$*?*

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http://dx.doi.org/10.1016/j.comgeo.2017.06.007
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Few showed that the length of a minimum spanning path of any n points in the unit square is at most $\sqrt{2n} + 7/4$ by a constructive proof: lay out about $\sqrt{n}$ equidistant horizontal lines, and then visit the points layer by layer, with the path alternating directions along the horizontal strips. An upper bound with a slightly better multiplicative constant for a path was derived by Karloff [13]. L. Fejes Tóth [4] had observed earlier that for n points of a regular hexagonal lattice in the unit square, the length of the minimum spanning path is asymptotically equal to $(4/3)^{1/4}\sqrt{n}$, where $(4/3)^{1/4} = 1.0745\ldots$.

By adapting the proof for minimum spanning path, Few also showed that the length of a minimum rectilinear Steiner tree of any n points in the unit square is at most $\sqrt{n} + 7/4$, that is, $s_{\llcorner}(n) \leq \sqrt{n} + 7/4$. Since $s(n) \leq s_{\llcorner}(n)$, this also yields the bound $s(n) \leq \sqrt{n} + 7/4$. Chung and Graham [2] reported an improved upper bound of $s(n) \leq 0.995\sqrt{n}$ (they gave details for an improvement to $0.99995\sqrt{n}$ only) and a lower bound (by the same example of a regular hexagonal lattice in the unit square) of $s(n) \geq (3/4)^{1/4}\sqrt{n} + O(1)$, where $(3/4)^{1/4} = 0.9306\ldots$. Note that $(4/3)^{1/4} = (3/4)^{1/4} \cdot 2/\sqrt{3}$, where $2/\sqrt{3}$ is the conjectured Steiner ratio in the plane.

In every dimension $d \geq 3$, Few [5] showed that the maximum length of a shortest path through n points in the unit cube is $\Theta(n^{1-1/d})$, and that the maximum length of a minimum rectilinear Steiner tree of n points in the unit cube is $O(n^{1-1/d})$. Snyder [18,19] then proved an asymptotically tight bound of $\Theta(n^{1-1/d})$ for the latter problem, extending the work of Few [5] and Chung and Graham [2]. Among others, some pieces of early work on the geometric variants of the Steiner tree problem (EST or RST) are [3,6–8,12,20,21]. Both variants are known to be NP-complete [9]. More recent developments and other problems in geometric networks can be found in [15].

In this paper we study the *rectilinear* version. Chung and Graham [2] observed that $s_{\llcorner}(n) \geq \sqrt{n} + O(1)$ is implied by subsets of points from a suitable square lattice and, in particular, $s_{\llcorner}(k^2) \geq k + 1$ for $k \geq 2$. They also reported the upper bound $s_{\llcorner}(n) \leq \sqrt{n} + 1 + o(1)$, in particular, $s_{\llcorner}(k^2) \leq k + 1$ for $k \geq 2$.

Here we revisit the problem and show that the argument in [2] justifying this upper bound does not stand. Consequently, both the asymptotic upper bound $s_{\llcorner}(n) \leq \sqrt{n} + 1 + o(1)$ and the upper bound $s_{\llcorner}(k^2) \leq k + 1$ for $k \geq 2$ remain without proof. We then revise the argument and obtain an upper bound of $s_{\llcorner}(n) \leq \sqrt{n} + 5/4 + o(1)$ and establish that indeed, $s_{\llcorner}(n) = \sqrt{n} + 1$ for $n = k^2$, for $k \geq 2$, and so this formula holds for an infinite sequence of values of n.

The fact that the upper bound $s_{\llcorner}(n) \leq \sqrt{n} + 1 + o(1)$ does not follow from the argument in [2] was also noticed by Brenner and Vygen [1] in their elaborate study of planar networks with respect to various criteria of comparison. They also deduced the upper bound $s_{\llcorner}(n) \leq \sqrt{n} + 3/2 + o(1)$ [1, Corollary, p. 130] without, however, delving into the gaps of the argument in [2]. After an in-depth discussion of the matter, we give a new approach and a further reduction in the constant additive term in our Theorem 1.

Chung and Graham [2] also conjectured that $s_{\llcorner}(n) \leq \sqrt{n} + 1$ for all $n \geq 2$. Here we reduce their conjecture to a number-theoretic conjecture regarding integer partitions that we propose. We then verify the new conjecture for small values of n, and for a *new* infinite sequence of values of n. In Section 2 we prove the following.

Theorem 1.

(i) *For every $n \geq 2$ we have $s_{\llcorner}(n) \leq \sqrt{n} + 5/4 + o(1)$.*
(ii) *For every $n \leq 50$ we have $s_{\llcorner}(n) \leq \sqrt{n} + 1$.*
(iii) *We also have: $s_{\llcorner}(k^2) \leq s_{\llcorner}(k^2+1) \leq k+1$, $s_{\llcorner}(k^2+2) \leq k+1+(k+1)^{-1}$, $s_{\llcorner}(k^2+k) \leq k + \frac{1}{2} - \frac{1}{2k+1}$, $s_{\llcorner}(k^2+k+1) \leq k+1/2$, and $s_{\llcorner}(k^2+2k) \leq k+1-\frac{1}{2k}$, for every $k \geq 2$; so in particular, the bound $s_{\llcorner}(n) \leq \sqrt{n}+1$ also holds for such n.*

We think that our proof method that establishes parts (ii) and (iii) could work for every n, but so far we have not been able to formulate a general argument. A combinatorial formulation and a relevant conjecture are proposed in Section 4.

2. Methods of proof

Our proofs are constructive and bear some similarities to earlier proofs. The first method of proof is geometric; it yields part (i) of Theorem 1. The second method of proof is purely combinatorial; it yields parts (ii) and (iii) of Theorem 1. We next present these two methods.

Few's method. In Few's proof [5] (as presented in a simpler but equivalent way by Chung and Graham [2]), the square U is subdivided into s strips by $s-1$ equally spaced horizontal segments $\ell_1, \ldots, \ell_{s-1}$ of unit length. Including the lower and upper sides of U yields $s+1$ horizontal unit segments $\ell_0, \ell_1, \ldots, \ell_{s-1}, \ell_s$. Then each point is in some strip (if a point is on the horizontal segment shared by two adjacent strips, associate the point with either strip, arbitrarily). For each point in some strip, take a vertical segment through the point to join it with the two horizontal segments bounding the strip. Take also the left and right sides of U. Then the n points are connected by $s+1$ horizontal and 2 vertical segments of length 1, and n vertical segments of length $1/s$. The total length of these segments is

$$(s+1) + 2 + n/s = (s + n/s) + 3. \tag{1}$$

In particular, the vertical segment of length $1/s$ through each point in a strip is the union of two vertical segments joined at the point, one connecting the point to a horizontal segment with an odd index, and the other connecting the point to a horizontal segment with an even index. From these segments we can construct two disjoint Steiner trees for the n points: