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On the $\mathcal{O}_\beta$-hull of a planar point set [☆,☆☆]

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ABSTRACT

We study the $\mathcal{O}_\beta$-hull of a planar point set, a generalization of the Orthogonal Convex Hull where the coordinate axes form an angle β. Given a set P of n points in the plane, we show how to maintain the $\mathcal{O}_\beta$-hull of P while β runs from 0 to π in $\Theta(n \log n)$ time and $O(n)$ space. With the same complexity, we also find the values of β that maximize the area and the perimeter of the $\mathcal{O}_\beta$-hull and, furthermore, we find the value of β achieving the best fitting of the point set P with a two-joint chain of alternate interior angle β.

1. Introduction

Let $\mathcal{O}_\beta$ be a set of two lines with slopes 0 and $\tan(\beta)$, where $0 < \beta < \pi$. A region in the plane is said to be $\mathcal{O}_\beta$*-convex*, if its intersections with all translations of any line in $\mathcal{O}_\beta$ are either empty or connected. An $\mathcal{O}_\beta$*-quadrant* is a translation of one of the ($\mathcal{O}_\beta$-convex) open regions that result from subtracting the lines in $\mathcal{O}_\beta$ from the plane. We call the quadrants of $\mathcal{O}_\beta$ as *top-right*, *top left*, *bottom-right*, and *bottom-left* according to their position with respect to the elements of $\mathcal{O}_\beta$, see Fig. 1(a). Let P be a set of n points, and $\mathcal{Q}$ the set of all $\mathcal{O}_\beta$-quadrants that are P-*free*; i.e., that contain no elements of P. The $\mathcal{O}_\beta$*-hull* of P is the set

$$\mathcal{O}_\beta\mathcal{H}(P) = \mathbb{R}^2 - \bigcup_{q \in \mathcal{Q}} q$$

of points in the plane belonging to all connected supersets of P which are $\mathcal{O}_\beta$-convex [3,11]. See Fig. 1(b).

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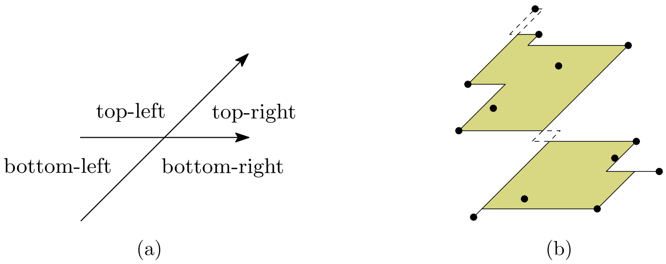

Fig. 1. (a) A set $\mathcal{O}_\beta$-hull, the top-right, top-left, bottom right, and bottom left quadrants. (b) The corresponding $\mathcal{O}_\beta$-hull of a point set.

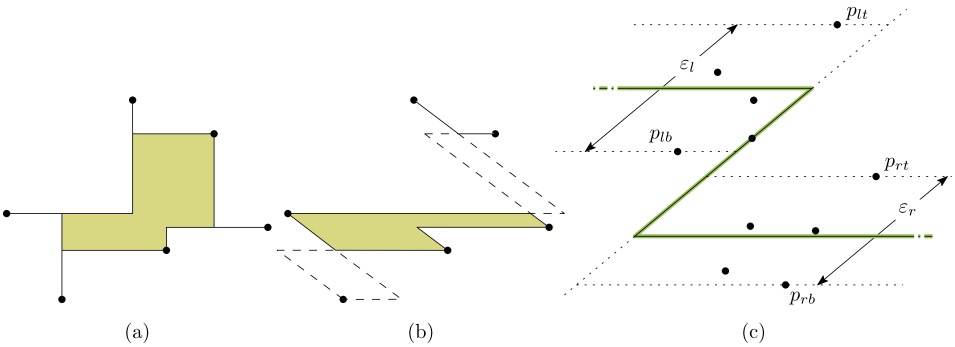

Fig. 2. (a) $\mathcal{O}_{\frac{\pi}{2}}\mathcal{H}(P)$. (b) $\mathcal{O}_{\beta_0}\mathcal{H}(P)$, where $\beta_0 > \frac{\pi}{2}$. (c) A two-joint non-orthogonal polygonal chain fitting a point set.

The concept of $\mathcal{O}_\beta$-convexity stemmed from the notion of *restricted orientations* [9], where geometric objects comply with a property (or a set of properties) related to some fixed set of lines. Researchers have extensively studied this notion by considering restricted-oriented polygons [9], proximity [18], visibility [17], and both restrictions and generalizations of $\mathcal{O}_\beta$-convexity. The particular case of *orthogonal convexity* [16] considers β to be fixed at $\frac{\pi}{2}$. In the more general *$\mathcal{O}$-convexity* [15,16], $\mathcal{O}_\beta$ is replaced by a (possibly infinite) set of lines with arbitrary orientations. Other restricted-oriented notions of convexity include *D-convexity* [8] and *$\mathcal{O}$-convexity* [14]. The former is based in a functional (rather than set-theoretical) definition, while the latter (unlike $\mathcal{O}_\beta$-convexity) always leads to connected sets. For a comprehensive compilation of studies on the area please refer to Fink and Wood [7]. Some recent computational results can be found in [1–3,12].

In this paper, we solve the problem of maintaining the combinatorial structure of $\mathcal{O}_\beta\mathcal{H}(P)$ while β goes from 0 to π, and apply this result to some optimization problems. Following the lines of Bae et al. [5], we find the values of β that maximize the area and the perimeter of $\mathcal{O}_\beta\mathcal{H}(P)$. In addition, we include an appendix extending the results from Díaz-Báñez et al. [6] to fit a two-joint not-necessarily orthogonal polygonal chain to a point set. See Fig. 2.

In all cases, our general approach is to perform an angular sweep. We first discretize the set $\{\beta : \beta \in (0, \pi)\}$ into a linear sequence of increasing angles $\{\beta_1, \beta_2, \ldots, \beta_{O(n)}\}$. While β runs from 0 to π, each β_i corresponds to an angle where there is a change in the combinatorial structure of $\mathcal{O}_\beta\mathcal{H}(P)$. We then solve the particular problem for any $\beta \in [\beta_1, \beta_2)$ in $O(n \log n)$ time, and show how to update our solution in logarithmic time in the subsequent intervals $[\beta_i, \beta_{i+1})$. All our algorithms run in $O(n \log n)$ time and $O(n)$ space.

Outline of the paper In Section 2 we show how to maintain the $\mathcal{O}_\beta$-hull of P while β goes from 0 to π. In Section 3 we extend this result to solve the optimization problems we mentioned above. We end in Section 4 with our concluding remarks.

2. The $\mathcal{O}_\beta$-hull of P

In this section we introduce definitions that are central to our results. We also show how to compute $\mathcal{O}_\beta\mathcal{H}(P)$ for a fixed value of β, and how to maintain its combinatorial structure while β runs from 0 to π.

2.1. Preliminaries

For the sake of simplicity, we will assume P to have no three colinear points, and no pair of points on a horizontal line. Consider the region $\mathcal{R}$ obtained by removing from the plane all top-right $\mathcal{O}_\beta$-quadrants free of elements of P. The *top-right $\mathcal{O}_\beta$-staircase* of P is the directed polygonal chain formed by the segment of the boundary of $\mathcal{R}$ that starts at the rightmost and ends at the topmost vertex (element of P that lies over the boundary) of $\mathcal{O}_\beta\mathcal{H}(P)$, with respect to the coordinate

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