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On the number of touching pairs in a set of planar curves

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ABSTRACT

Given a set of planar curves (Jordan arcs), each pair of which meets — either crosses or touches — exactly once, we establish an upper bound on the number of touchings. We show that such a curve family has $O(t^2 n)$ touchings, where t is the number of faces in the curve arrangement that contains at least one endpoint of one of the curves. Our method relies on finding special subsets of curves called quasi-grids in curve families; this gives some structural insight into curve families with a high number of touchings.

1. Introduction

The combinatorial examination of incidences in the plane has proven to be a fruitful area of research. The first seminal results are the crossing lemma that establishes a lower bound on the number of edge crossings in a planar drawing of a graph (Ajtai et al., Leighton [1,2]), and the theorem by Szemerédi and Trotter [3], concerning the number of incidences between lines and points. Soon, the incidences of more general geometric objects (segments, circles, algebraic curves, pseudo-circles, Jordan arcs, etc.) became the center of attention [4–9]. With the addition of curves, the distinction between touchings and crossings is in order.

Usually, the curves are either Jordan arcs, i.e., the image of an injective continuous function $\varphi : [0, 1] \to \mathbb{R}^2$, or closed Jordan curves, where φ is injective on $[0, 1)$ and $\varphi(0) = \varphi(1)$. Generally, it is supposed that the curves intersect in a finite number of points, and that the curves are in general position: three curves cannot meet at one point, and (in case of Jordan arcs) an endpoint of a curve does not lie on any other curve. (For technical purposes, we will allow curve endpoints to coincide in some proofs.)

Let P be a point where curve a and b meet. Take a circle γ with center P and a small enough radius so that it intersects both a and b twice, and the disk determined by γ is disjoint from all the other curves , and contains no other intersections of a and b. Label the intersection points of γ and the two curves with the name of the curve. We say that a and b *cross* in P if the cyclical permutation of labels around γ is *abab*, and a and b *touch* in P if the cyclical permutation of labels is *aabb*. In a family of curves, let X be the set of crossings and T be the set of touchings.

The Richter–Thomassen conjecture [10] states that given a collection of n pairwise intersecting closed Jordan curves in general position in the plane, the number of crossings is at least $(1 - o(1))n^2$. A proof of the Richter–Thomassen conjecture has recently been published by Pach et al. [11]. They show that the same result holds for Jordan arcs as well.

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It would be preferable to get more accurate bounds for the ratio of touchings and crossings. Fox et al. constructed a family of x-monotone curves with ratio $|X|/|T| = O(\log n)$ [12]. If we restrict the number of intersections between any two curves, then it is conjectured that the above ratio is much higher. It has been shown that a family of intersecting pseudo-circles (i.e., a set of closed Jordan-curves, any two of which intersect exactly once or twice) has at most $O(n)$ touchings [7]. We would like to examine a similar statement for Jordan arcs.

A family of Jordan arcs in which any pair of curves intersect at most once (apart from the endpoints) will be called a *family of pseudo-segments*. Our starting point is this conjecture of János Pach [13]:

Conjecture 1. *Let $\mathcal{C}$ be a family of pseudo-segments. Suppose that any pair of curves in $\mathcal{C}$ intersect exactly once. Then the number of touchings in $\mathcal{C}$ is $O(n)$.*

A family of pseudo-segments is *intersecting* if every pair of curves intersects (i.e., either touches or crosses) exactly once outside their endpoints.

Two important special cases of the above are the cases of *grounded* and *double-grounded* curves. (The definitions are taken verbatim from [9].) A collection $\mathcal{C}$ of curves is *grounded* if there is a closed Jordan curve g called *ground* such that each curve in $\mathcal{C}$ has one endpoint on g and the rest of the curve is in the exterior of g. The collection is *double grounded* if there are disjoint closed Jordan curves g_1 and g_2 such that each curve $c \in \mathcal{C}$ has one endpoint on g_1 and the other endpoint on g_2, and the rest of c is disjoint from both g_1 and g_2.

According to our knowledge the best upper bound is $O(n \log n)$ for the number of touchings in a double-grounded x-monotone family of pseudo-segments [14] and we do not know any (non-trivial) result for the grounded case.

1.1. Our contribution

Let $\mathcal{C}$ be an intersecting family of pseudo-segments. There is a planar graph drawing that corresponds to this family: the vertices are the crossings and touchings, and the edges are the sections of the curves between neighboring intersections. (Notice that the sections between curve endpoints and the neighboring intersections are not represented in this graph.) Consider the faces of this planar graph drawing. Let $t_{\mathcal{C}}$ be the number of faces that contain an endpoint of at least one curve in $\mathcal{C}$. Our main theorem can be stated as follows:

Theorem 2. *Let $\mathcal{C}$ be an n-element intersecting family of pseudo-segments on the Euclidean plane. Then the number of touchings between the curves is $f(n) = O(t_{\mathcal{C}}^2 n)$.*

If $t_{\mathcal{C}}$ is constant, this theorem settles Conjecture 1. Note that this includes the case when $\mathcal{C}$ is a double-grounded intersecting family of pseudo-segments:

Corollary 3. *Let $\mathcal{C}$ be an n-element double-grounded intersecting family of pseudo-segments. Then the number of touchings between the curves is $O(n)$.*

A careful look at the proof of the main theorem yields the following result for grounded intersecting families of pseudo-segments:

Theorem 4. *Let $\mathcal{C}$ be an n-element grounded intersecting family of pseudo-segments. Then the number of touchings between the curves is $O(t_{\mathcal{C}} n)$.*

The intuition behind our approach can be described as follows. Curves starting in the same face of an arrangement can be thought of as curves having the same endpoints. A curve going from point A to B that touches some other curve g can do that touching only in a constant number of ways, depending on which side of g is touched and in which direction. We observe that a collection of curves going from A to B must therefore contain a subcollection that touch g the same way, and these curves must have a very special grid-like structure, which we call *quasi-grids*.

It turns out that quasi-grids always emerge when we take two grid families of pseudo-segments, one containing curves from A to B, the other containing curves from C to D. Note that a curve touching all curves in a large quasi-grid has to lie outside the "grid cells", since it cannot cross the quasi-grid curves, and within a "grid cell" it could only reach at most four curves. If we find two curves touching the same large quasi-grid, then (intuitively) those two curves would have many intersections — this is not possible in an intersecting family of pseudo-segments. We show that the number of touchings between a pair of fixed endpoint curve families is linear in the size of these families. We then use this observation to get the bound on the total number of touchings.

2. Proof of the main theorem

The rigorous proof of our main theorem is based upon a key lemma. Its proof anticipates and uses several technical lemmas which are detailed in Sections 3 and 4.