Computational Geometry 67 (2018) 42–52

Contents lists available at ScienceDirect

Computational Geometry: Theory and Applications

www.elsevier.com/locate/comgeo

Continuous Yao graphs

Davood Bakhshesh [a], Luis Barba [b,1], Prosenjit Bose [c], Jean-Lou De Carufel [d,2], Mirela Damian [e,3], Rolf Fagerberg [f,4], Mohammad Farshi [a,*], André van Renssen [g,h,5], Perouz Taslakian [i,j], Sander Verdonschot [c]

[a] *Combinatorial and Geometric Algorithms Lab., Department of Computer Science, Yazd University, Islamic Republic of Iran*
[b] *Department of Computer Science, ETH Zürich, Switzerland*
[c] *School of Computer Science, Carleton University, Canada*
[d] *School of Electrical Engineering and Computer Science, University of Ottawa, Ottawa, Canada*
[e] *Department of Computing Sciences, Villanova University, United States*
[f] *Department of Mathematics and Computer Science, University of Southern Denmark, Denmark*
[g] *National Institute of Informatics, Tokyo, Japan*
[h] *JST, ERATO, Kawarabayashi Large Graph Project, Japan*
[i] *School of Science and Engineering, American University of Armenia, Armenia*
[j] *Element AI, Canada*

ARTICLE INFO

Article history:
Received 23 December 2016
Received in revised form 27 May 2017
Accepted 14 September 2017
Available online 16 October 2017

Keywords:
Yao graph
Spanner
Spanning ratio
Self-approaching graph

ABSTRACT

In this paper, we introduce a variation of the well-studied Yao graphs. Given a set of points $S \subset \mathbb{R}^2$ and an angle $0 < \theta \leqslant 2\pi$, we define the *continuous Yao graph* $cY(\theta)$ with vertex set S and angle θ as follows. For each $p, q \in S$, we add an edge from p to q in $cY(\theta)$ if there exists a cone with apex p and aperture θ such that q is a closest point to p inside this cone.

We study the spanning ratio of $cY(\theta)$ for different values of θ. Using a new algebraic technique, we show that $cY(\theta)$ is a spanner when $\theta \leqslant 2\pi/3$. We believe that this technique may be of independent interest. We also show that $cY(\pi)$ is not a spanner, and that $cY(\theta)$ may be disconnected for $\theta > \pi$, but on the other hand is always connected for $\theta \leqslant \pi$. Furthermore, we show that $cY(\theta)$ is a region-fault-tolerant geometric spanner for convex fault regions when $\theta < \pi/3$. For half-plane faults, $cY(\theta)$ remains connected if $\theta \leqslant \pi$. Finally, we show that $cY(\theta)$ is not always self-approaching for any value of θ.

* Corresponding author.

E-mail addresses: dbakhshesh@gamil.com (D. Bakhshesh), luis.barba@inf.ethz.ch (L. Barba), jit@scs.carleton.ca (P. Bose), jdecaruf@uottawa.ca (J.-L. De Carufel), mirela.damian@villanova.edu (M. Damian), rolf@imada.sdu.dk (R. Fagerberg), mfarshi@yazd.ac.ir (M. Farshi), andre@nii.ac.jp (A. van Renssen), ptaslakian@aua.am (P. Taslakian), sander@cg.scs.carleton.ca (S. Verdonschot).

[1] Research partially funded by the ETH Postdoctoral Fellowship.
[2] Research funded by NSERC.
[3] Research supported by NSF grant CCF-1218814.
[4] Partially supported by the Danish Council for Independent Research Natural Sciences, grant DFF-1323-00247.
[5] Supported by JST ERATO Grant Number JPMJER1201, Japan.

https://doi.org/10.1016/j.comgeo.2017.10.002

1. Introduction

Let S be a set of points in the plane. The complete geometric graph with vertex set S has a straight-line edge connecting each pair of points in S. Because the complete graph has quadratic size in terms of number of edges, several methods for "approximating" this graph with a graph of linear size have been proposed.

A geometric t-spanner G of S is a spanning subgraph of the complete geometric graph of S with the property that for all pairs of points p and q of S, the length of the shortest path between p and q in G is at most t times the Euclidean distance between p and q denoted by $|pq|$.

The *spanning ratio* of a spanning subgraph is the smallest t for which this subgraph is a t-spanner. For a comprehensive overview of geometric spanners and their applications, we refer the reader to the book by Narasimhan and Smid [16].

A simple way to construct a t-spanner is to first partition the plane around each point $p \in S$ into a fixed number of cones[6] and then add an edge connecting p to a closest vertex in each of its cones. These graphs have been independently introduced by Flinchbaugh and Jones [11] and Yao [19], and are referred to as *Yao graphs* in the literature. It has been shown that Yao graphs are good approximations of the complete geometric graph [7,3,6,5,8,10,4].

We denote the Yao graph defined on S by Y_k, where k is the number of cones, each having aperture $\theta = 2\pi/k$. Clarkson [7] was the first to remark that Y_{12} is a $(1+\sqrt{3})$-spanner in 1987. Althöfer et al. [3] showed that for every $t > 1$, there is a k such that Y_k is a t-spanner. For $k > 8$, Bose et al. [6] showed that Y_k is a geometric spanner with spanning ratio at most $1/(\cos\theta - \sin\theta)$. This was later strengthened to show that for $k > 6$, Y_k is a $1/(1 - 2\sin(\theta/2))$-spanner [5]. Damian and Raudonis [8] proved a spanning ratio of 17.64 for Y_6, which was later improved by Barba et al. to 5.8 [4]. The same authors also improved the spanning ratio of Y_k for all odd values of $k \geqslant 5$ to $1/(1 - 2\sin(3\theta/8))$ [4]. In particular, they showed an upper bound on the spanning ratio for Y_5 of $2 + \sqrt{3} \approx 3.74$. Bose et al. [5] showed that Y_4 is a 663-spanner. For $k < 4$, El Molla [10] showed that there is no constant t such that Y_k is a t-spanner.

Yao graphs are based on the implicit assumption that all points use identical cone orientations with respect to an extrinsic fixed direction. From a practical point of view, if these points represent wireless devices and edges represent communication links for instance, the points would need to share a global coordinate system to be able to orient their cones identically. Potential absence of global coordinate information adds a new level of difficulty by allowing each point to spin its cone wheel independently of the others. In this paper we take a first step towards reexamining Yao graphs in light of intrinsic cone orientations, by introducing a new class of graphs called *continuous Yao graphs*.

Given an angle $0 < \theta \leqslant 2\pi$, the continuous Yao graph with angle θ, denoted by $cY(\theta)$, is the graph with vertex set S, and an edge connecting two points p and q of S if there exists a cone with angle θ and apex p such that q is a closest point to p inside this cone. In contrast with the classical construction of Yao graphs, for the continuous version the orientation of the cones is arbitrary. One can imagine rotating a cone with angle θ around each point $p \in S$ and connecting it to each point that becomes closest to p inside the cone during this rotation. To simplify our proofs we assume *general position*, in the sense that no two points lie at the same distance from any point in S.

In contrast with the Yao graph, the continuous Yao graph has the property that $cY(\theta) \subseteq cY(\gamma)$ for any $\theta \geqslant \gamma$. This property provides consistency as the angle of the cone changes and could be useful in potential applications requiring scalability. Another advantage of continuous Yao graphs over regular Yao graphs is that they are invariant under rotations of the input point set. However, unlike Yao graphs that guarantee a linear number of edges, continuous Yao graphs may have a quadratic number of edges in the worst case. (Imagine, for instance, the input points evenly distributed on two line segments that meet at an angle $\alpha < \pi$. For any $\theta < \alpha$, $cY(\theta)$ includes edges connecting each point on one line segment to each point on the other line segment.)

Before summarizing our results, we introduce two more definitions. Let G be a geometric graph with vertex set S. For any pair of vertices $s, t \in S$, a path from s to t in G is called *self-approaching* if, for every point q on the path (not necessarily a vertex), a point moving continuously on the path from s to q never gets further away from q. The graph G is *self-approaching* if it contains a self-approaching path between every pair of vertices.

For any integer $q > 0$, a geometric graph H is a *q-fault tolerant* t-spanner for S if, for any subset $S' \subseteq S$ with $|S'| \leq q$, the graph $H \setminus S'$ is a t-spanner of $S \setminus S'$. Li et al. [14] extend the Yao structure Y_k to a new structure $Y_{k,q}$ as follows: in each cone with apex $a \in S$, add edges connecting a to $q+1$ closest nodes. The authors show that, for $k > 6$, the graph $Y_{k,q}$ is a q-fault tolerant t-spanner of S, for some constant $t > 0$. Wang et al. [18] use a similar method to construct q-fault tolerant graphs in 3D. For any region F in the plane, we define $G \ominus F$ to be the remaining graph after removing all vertices of G that lie inside F and all edges of G that intersect F. Given a set $\mathcal{F}$ of regions in the plane, we say that G is an *$\mathcal{F}$-fault tolerant* t-spanner if, for any region $F \in \mathcal{F}$, the graph $G \ominus F$ is a t-spanner for $K_S \ominus F$, where K_S is the complete geometric graph on S. To the best of our knowledge, there are no results on region-fault tolerance of Yao graphs.

In this paper we study three properties of continuous Yao graphs: the spanning property, the self-approaching property and the region-fault tolerance property. In Section 2, we show that $cY(\theta)$ has spanning ratio at most $1/(1 - 2\sin(\theta/4))$ when $\theta < 2\pi/3$. However, the argument used in this section breaks when $\theta = 2\pi/3$. To deal with this case, we introduce a new algebraic technique based on the description of the regions where induction can be applied. To the best of our knowledge, this is the first time that algebraic techniques are used to bound the spanning ratio of a graph. As such, our

[6] The orientation of the cones is the same for all vertices and each cone is half-closed: it includes its clockwise boundary.