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Improved near-exact distributions for the product of independent Generalized Gamma random variables

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ABSTRACT

The Generalized Gamma distribution is an important distribution in Statistics since it has as particular cases many well known and important distributions and also due to its very interesting modeling properties, which makes it an attractive tool. The distribution of the product of independent Generalized Gamma distributions is investigated. Most of the results available for this distribution are based on Meijer-G or H functions which may still be very difficult to handle. Therefore, near-exact distributions which are based on the Generalized Near-Integer Gamma distribution and which have density and cumulative distribution functions easily implementable and computationally appealing are developed. Numerical studies with computationally intensive analyses are carried out to study the accuracy of these approximations in different scenarios. Also computational modules are provided for the implementation of these approximations. Finally, an example of application to quality control in microbiology is provided.

1. Introduction

The Generalized Gamma distribution, introduced in Stacy (1962), is an important tool for modeling problems in Statistics mainly due to its flexibility. Applications can be found in different areas of research, such as Physics, Econometrics, Wireless Communications, Reliability Analysis, Hydrological Processes and Life Testing Models. For example, in Aalo (2005) the Generalized Gamma distribution is used to characterize both multipath and shadow fading processes in wireless communication systems, in Smirnov (2008) an approximation, based on the Generalized Gamma distribution, of the real line shape of a scintillation detector is considered, and in Zaninetti (2010) we can find an interesting application in modeling the luminosity function of galaxies. Another point of interest of this distribution is the fact of having as particular cases some of the most important and known distributions in Statistics such as the Gamma, Weibull and Rayleigh distributions, for more details see for example Coelho and Mexia (2010). Results on inference on the parameters of the Generalized Gamma distribution can be found in Gomes et al. (2008), Hager and Bain (1970), Harter (1967), Lawless (1980), Parr and Webster (1965) and Stacy and Mihram (1965).

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The product of independent Generalized Gamma random variables arises naturally in the problems posed in the above mentioned fields of research, for example, in Ali et al. (2008) the authors consider an application to drought data from Nebraska; they use the product of two Generalized Gamma random variables as the distribution of the magnitude of droughts. In Zaninetti (2008) the author uses the product of two Gamma random variables in problems related with the luminosity function for galaxies, and in Karagiannidis et al. (2006) the authors use these distributions to study the outage probability and the average bit-error probability for phase and frequency-modulated signaling. In this last work the product of independent Rayleigh random variables is addressed. Furthermore, in Coelho and Arnold (2014) the authors mention that the product of Generalized Gamma distributions has also as particular cases the exact distribution of the generalized variance, the exact distribution of discriminants or Vandermonde determinants and the distribution of the product of any power of the absolute value of independent Normal random variables. In Marques et al. (2015) several interesting applications are illustrated for the linear combination of Gumbel random variables which may be obtained by a trivial transformation of the product of independent Generalized Gamma random variables. In Section 4 it is given an example of application to quality control in microbiology.

However the exact distribution of the product of independent Generalized Gamma random variables does not have a simple expression which makes almost impossible its use in practical terms. Most of the existing approaches are based on Meijer-G or H functions which may still be very difficult to use, this was already stated by other authors in different studies Burda et al. (2012), Coelho and Arnold (2014) and Marques et al. (2015). Therefore new, easy to implement and extremely precise near-exact approximations are proposed for the product of independent Generalized Gamma random variables which, given its precision, can be used instead of the exact distribution. The approximations developed have as starting point the methodology used in Marques (2012) and Marques et al. (2015) however through a new and innovative procedure it is possible to improve and generalize the results obtained in Marques et al. (2015) for the linear combination of Gumbel random variables and in Coelho and Arnold (2014) and Marques (2012) for the product of independent Generalized Gamma random variables.

The exact distribution was first studied in Mathai (1972), using the inverse Mellin transform and H-functions, however these functions have very complicated expressions and are not implemented in most of the softwares available. This author also presents representations in terms of Meijer-G functions for two particular cases; when all the parameters β_j, in (1) of Section 2.1, are equal or when these are rational numbers, still even in these cases the Meijer-G functions can only be used in extremely limited cases which restrains the practical usefulness of these results. At the same time Podolski (1972) developed a representation in terms of Meijer-G function for the distribution of the product of independent Generalized Gamma random variables when all the β_j are equal. In Ali et al. (2008) and Malik (1968) the exact density and cumulative distribution functions are obtained, for the case of the product of only two Generalized Gamma random variables, using the modified Bessel function. Some results have been obtained in Karagiannidis et al. (2006) for the product of Generalized Gamma random variables and in Salo et al. (2006) and Salo et al. (2004) for the product of independent Rayleigh distributions, however these results are also based on the Meijer-G function and as such, according to what has already been mentioned, are difficult to use. In Coelho and Arnold (2014) the authors have developed near-exact distributions for the product of independent Generalized Gamma random variables using a different approach which is based on truncations of an infinite representation of the Gamma function. However it is shown that with a simple, new and general approach it is possible to improve the results obtained in Coelho and Arnold (2014), and also to address, with high precision, more complex scenarios.

The methodology used is as follows; using the characteristic function of the logarithm of the product of independent Generalized Gamma random variables, a different representation for the exact distribution is presented and, based on this representation, first it was developed a simple and accurate near-exact distribution for the product of independent Generalized Gamma random variables, which can be easily implemented and used in practical terms Marques (2012). Then, using this first representation as a basis, new near-exact distributions were developed which are even more accurate and as such can be used in problems and applications where an extremely high precision is needed. In Section 2 it is shown that the exact distribution of the logarithm of the product of independent Generalized Gamma random variables may be written as the distribution of the sum of two independent random variables, one with the distribution of the sum of independent logGamma distributions multiplied by a parameter and the other with the distribution of a shifted Generalized Integer Gamma (SGIG) distribution Coelho (1998) and Marques et al. (2015). Based on this result simple but highly accurate near-exact distributions were developed which are based on the shifted Generalized Near-Integer Gamma (SGNIG) distribution Coelho (2004) and Marques et al. (2015), or on mixtures of this distribution. By construction, a new parameter, γ, is introduced into the previous representations which allows the improvement of the quality of the near-exact approximations by adjusting its value; it is shown that the approximations become more precise for high values of γ. The near-exact distributions are developed in such way that they will have the same first m^* moments as the exact distribution, and as such the performance of these near-exact approximations can also be improved by raising the value of m^*. In Section 3 a measure of proximity between characteristic functions is used, that is also an upper bound on the proximity between the corresponding distribution functions, to assess the quality of the near-exact distributions developed. In addition, simulations are carried out that show the accuracy of the near-exact quantiles. Plots of the density functions of the three scenarios considered and quantile–quantile plots with the exact and near-exact quantiles are also presented. In Section 4 one provides an example of application to quality control in microbiology, where the results developed are used to perform external quality control in the field of environment analyses. A method called "proficiency testing" is considered. In Section 5, computational modules for the implementation of the results developed are provided. Section 6 is dedicated to conclusions.