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Nonlinear expectile regression with application to Value-at-Risk and expected shortfall estimation

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A B S T R A C T

This paper considers nonlinear expectile regression models to estimate conditional expected shortfall (ES) and Value-at-Risk (VaR). In the literature, the asymmetric least squares (ALS) regression method has been widely used to estimate expectile regression models. However, no literatures rigorously investigated the asymptotic properties of the ALS estimates in nonlinear models with heteroscedasticity. Motivated by this aspect, this paper studies the consistency and asymptotic normality of the ALS estimates and conditional VaR and ES in those models. To illustrate, a simulation study and real data analysis are conducted.

1. Introduction

Market risk means the risk of losses in financial position induced from movements in market prices such as stock and bond prices, exchange rates, and commodity prices. During the past decades, financial market scale has grown explosively and many kinds of financial commodities such as derivatives have been designed through academic innovation, technological developments and worldwide deregulation. Subsequently, new market risk measures were demanded to measure the risk of high complex financial commodities. McNeil et al. (2005) commented that these changes of financial market make a solid background on the birth of Value-at-Risk (VaR), which is now one of the most popular standard risk measures in finance industry. The VaR is defined as the maximum potential loss of portfolio over given holding period at a fixed confidence level and is expressed as the tail quantile of the portfolio return distribution. Despite of its merit, such as an ease at its usage, the VaR is well known to have some shortcomings such as the failure of sub-additivity, that is, the risk dispersion effect is dissatisfied: see Yamai and Yoshiba (2002), and further, it provides no information about the tail part beyond VaR. To circumvent these disadvantages, practitioners alternatively proposed to use expected shortfall (ES), defined as the conditional expectation of the losses beyond VaR: see Artzner et al. (1999). Together with VaR, ES is widely accepted as a useful risk measure in finance industry: see, for instance, Gerlach and Chen (in press) for some recent developments of ES in financial time series.

The estimation method of VaR falls into three categories (cf. Manganelli and Engle, 2004). The first is the parametric method wherein a return process of interest is generated by a model with error terms following a specific distribution, for example, linear GARCH models with normal innovations. In this case, one estimates all the parameters and VaR based on the given parametric model; the second is the nonparametric method that uses return data with no assumptions about a model. In this case, VaR is estimated as the quantile of a moving window, while ES is estimated as the average of values

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beyond the estimated VaR; the third is the semiparametric method that includes the extreme value theory (EVT) and quantile regression approach. Among these, here we focus on the semiparametric method based on quantile regression approach. In fact, Engle and Manganelli (2004) proposed CAViaR model and used the quantile regression method to estimate model parameters. The CAViaR model comprises of the part of a conditional VaR model and the error terms with their conditional quantiles set to be zero at a given level. Owing to this simple structure, the parameter estimation becomes more feasible to implement. In analogy of CAViaR model, Taylor (2008) proposed conditional autoregressive expectile (CARE) models and used the asymmetric least squares (ALS) regression method to obtain ES (cf. Newey and Powell, 1987). Taylor (2008) empirically demonstrated the validity of his method in various models, but to our knowledge, no literatures have investigated the asymptotic properties of the ALS estimates in CARE models. Motivated by this, we are led to study the consistency and asymptotic normality of ALS estimates in more general expectile regression models accommodating CARE models. To demonstrate the validity of our method, we conduct a simulation study for the expectile regression models induced from several GARCH type models and evaluate the performance of the VaR and ES estimation. Further, we analyze the stock prices of Hyundai Motors and Randgold Resources Limited (GOLD), a gold company, by fitting a linear GARCH(1,1) model and the Glosten et al. (1993) (JGR) model, respectively.

The remainder of this paper is organized as follows. In Section 2, we verify the consistency and asymptotic normality of ALS estimates in nonlinear expectile regression models. In Section 3, we conduct a simulation study and real data analysis for illustration. In Section 4, we provide the proofs for the results obtained in Section 2.

2. Asymptotic properties of ALS estimates

For a given univariate random variable Y, provided $E|Y| < \infty$, the τth expectile of Y is defined by

$$\mu(\tau) = \arg\min_m E\left[\rho_\tau(Y - m) - \rho_\tau(Y)\right],$$

where $\tau \in (0, 1)$ and $\rho_\tau(x) = |\tau - I(x < 0)|x^2$ is an asymmetric least squares loss function. Newey and Powell (1987) pointed out that the τth expectile is determined by the expectation of the tail part of the given random variable, that is,

$$\left(\frac{1-2\tau}{\tau}\right)E[(Y - \mu(\tau))I(Y < \mu(\tau))] = \mu(\tau) - E(Y).$$

Taylor (2008) used this property as a key to determine the relationship between the expectile and ES for $\theta \in (0, 1)$ as follows:

$$ES(\theta) = \left(1 + \frac{\tau}{(1-2\tau)\theta}\right)\mu(\tau) - \frac{\tau}{(1-2\tau)\theta}E(Y), \tag{2.1}$$

where $0 < \tau < 1$ is a real number satisfying $\mu(\tau) = VaR(\theta)$, the Value-at-Risk of Y at θ. This relationship explicitly shows how the expectile is linked with ES through VaR: see also (2.7) to exhibit the relationship between their conditional versions in time series models.

Aigner et al. (1976) proposed the asymmetric least squares (ALS) regression for the linear model:

$$Y_t = X_t^T\beta^o + \epsilon_t, \tag{2.2}$$

where $X_t = (1, x_1, \ldots, x_p)^T$, β^o is a true parameter, and ϵ_t are appropriately designed error terms depending upon the expectile level τ. Later, using the location and scale equivalent property of expectiles, Newey and Powell (1987) showed that the τth conditional expectile of Y_t given X_t in Eq. (2.2) is given by

$$\mu_{Y_t}(\tau) = X_t^T\beta^o + \mu_{\epsilon_t}(\tau),$$

where $\mu_{\epsilon_t}(\tau)$ is the τth expectile of ϵ_t. Further, if ϵ_t are i.i.d. and $\mu(\tau)$ is their τth conditional expectile, the τth conditional expectile of Y_t is expressed as $\mu_{Y_t}(\tau) = X_t^T\beta(\tau)$, where $\beta(\tau) = \beta^o + \mu(\tau)e_1$ and $e_1 = (1, 0, \ldots, 0)'_{p+1}$. Based on this expression, they defined the ALS estimator of $\beta(\tau)$ by

$$\hat\beta_n(\tau) = \arg\min_\beta \sum_{t=1}^n \rho_\tau(Y_t - X_t^T\beta),$$

investigated the asymptotic properties of the above ALS estimator, and proposed a test for heteroskedasticity and asymmetry for model (2.2). Although their result is extendable to other models, to our knowledge, no literatures have considered this issue in nonlinear regression models. Motivated by this, we are led to study the asymptotic properties of the ALS estimator for a class of nonlinear models with heteroscedasticity that particularly accommodate the CARE models proposed by Taylor (2008).

Let us consider the expectile regression model:

$$\begin{aligned} Y_t &= f(Y_{t-1}, Y_{t-2}, \ldots; \beta^o) + \epsilon_{t\tau} \\ &\equiv f_t(\beta^o) + \epsilon_{t\tau}, \quad 0 < \tau < 1, \end{aligned} \quad t \in \mathbb{Z}, \tag{2.3}$$