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Behavior of EWMA type control charts for small smoothing parameters

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ABSTRACT

A general family of EWMA charts is considered for monitoring an arbitrary parameter of the target process. The distribution of the run length is analysed for the case when the smoothing parameter tends to zero. The key impact on the results from the use of the exact variance of the control statistics vs. the asymptotic one and the presence of a head start. For fixed head start, the run lengths for both the exact and asymptotic monitoring procedures degenerate to a binary quantity. To guarantee a feasible monitoring procedure, the head start has to be chosen proportional to the smoothing parameter and the control statistics have to be modified when used with the asymptotic variance. This result underlines the weakness of schemes with a fixed head start and of schemes based on the asymptotic variance if the smoothing parameter is small. The assumptions on the target process are very weak, and are usually satisfied for stationary processes. In addition, the asymptotic equivalence of the EWMA schemes and of repeated significance tests is shown.

1. Introduction

The exponentially weighted moving average (EWMA) chart of Roberts (1959) is one of the most popular approaches to statistical surveillance. Its main advantage, especially valued by practitioners, is its simple and intuitive implementation based on the exponential smoothing of the historical values. Similar to the CUSUM chart of Page (1954), the smoothing parameter regulates the impact of past observations. Setting it equal to one leads to the Shewhart chart, cf. Shewhart (1931). Smaller values of the smoothing parameter lead to an increasing influence of the preceding observations. The EWMA scheme is mainly applied for monitoring the mean of independent data, but the procedure can be adapted in a straightforward way to monitor any characteristics of the process. The stochastic structure of the data generating process can be rather general too. A chart for the standard deviation of an independent random process was introduced by Crowder and Hamilton (1992). The extension of the EWMA chart to monitor the mean of a stationary time series was given by Schmid (1997). EWMA charts for the variance of a stationary process were proposed in Schipper and Schmid (2001) while Rosołowski and Schmid (2003) discussed simultaneous EWMA schemes for the mean, the variances, and the autocovariances.

The choice of the optimal smoothing parameter is in general an unsolved problem. Several authors have given recommendations for specific setups of the monitoring problem. Montgomery (2009) states that values between 0.05 and 0.25 work well in practice. Lucas and Saccucci (1990) determine, via simulations, the optimal value of the smoothing parameter which minimizes the out-of-control average run length (ARL) for a given value of the expected shift in the

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target process and for a given in-control ARL. The authors rely on a grid search over the interval [0.03;1]. The non-zero lower boundary in the study is due to numerical instabilities arising when the smoothing parameter approaches zero (see, e.g., Brook and Evans, 1972, Crowder, 1987).

Recent technological developments allow showing that the out-of-control ARL is decreasing in the smoothing parameter, implying that the limiting chart with the smoothing parameter approaching zero in the limit should be analysed more thoroughly. This problem was described by Chan and Zhang (2000). Morais et al. (forthcoming) established several important limit results for the distribution of the run length of the one-sided EWMA chart for the mean, assuming that the target process is independent and normally distributed. They proved that the in-control ARL of the EWMA scheme based on the exact variance is a decreasing function in the smoothing parameter, and showed that the limiting chart as the smoothing parameter approaches zero is equivalent to the repeated significance test. Note that both papers focus on the mean chart and independent data.

In the present paper we consider a very general family of EWMA schemes which can be used to monitor an arbitrary real-valued parameter of a stationary process. Because of its generality, this approach covers most of the EWMA schemes discussed in the literature. Furthermore, the suggested approach allows us to consider one-sided and two-sided charts within a single framework. We distinguish between the charts based on the exact variance, i.e., the control limits change in time, and the charts based on the asymptotic variance, i.e., the control limits are fixed. Since in the former case the variance must be recalculated at each time point, most practitioners prefer to work with the asymptotic variance.

A popular approach in modern statistical process control is to use a head start, i.e., to set the starting value of the control statistics not equal to the target value of the parameter. If correctly selected, the head start helps to improve the performance of the charts. We show in Section 3 that the run lengths of the charts based on the exact variance converge in distribution to the run length of the repeated significance test if the head start is proportional to the smoothing parameter. If the head start is constant, as it usually is, then the control chart degenerates, i.e., it gives either always or never signals. In the most realistic case, the probability of a signal up to a fixed point in time converges to 1 as well in the in-control state as in the out-of-control state. This is a very unpleasant property. Moreover, it shows that the ARL converges to infinity if the smoothing parameter tends to zero. For the EWMA scheme based on the asymptotic variance, an additional normalization of the procedure is required. Section 4 contains illustrative examples, which explain the limiting performance and the damaging impact of a constant head start.

2. Modified EWMA charts for stationary processes

In the following, we consider a very general surveillance problem. We make use of an EWMA type control scheme. Suppose that observations $x_1, x_2, \ldots$ of the real-valued stochastic process $\{X_t\}$ defined on the probability space $(\Omega, \mathcal{F}, P)$ are sequentially taken. Suppose that the distribution of $(X_1, \ldots, X_t)$ depends on a real-valued parameter θ_t. For instance, θ_t could be the mean of the process, its variance, etc. Let $\theta_t = \theta$ for all $t = 1, \ldots, q-1$ and $\theta_t \neq \theta$ for $t \geq q$. Here θ is assumed to be a known value. The process $\{X_t\}$ is said to be in-control if $q = \infty$ and we refer to the corresponding process as the in-control process. If $q < \infty$, then $\{X_t\}$ is said to be out of control. Note that here we do not make any further assumption on $\{X_t\}$.

Our aim is to detect whether the parameter changes over time or whether it is constant. In order to quickly detect the change, we use a sequential procedure and after every new observation, analyse whether a change has occurred or not.

Suppose that T_t is a point estimator of θ_t based on the information set available at time point t, i.e. $T_t = f_t(X_1, \ldots, X_t)$ with f_t a measurable function. Assume that T_t is an unbiased estimator of θ in the in-control state, i.e. $E(T_t) = \theta$ for all $t \geq 1$ provided that $q = \infty$. If θ_t is the mean of $\{X_t\}$, then we can choose, e.g., $T_t = X_t$ (Schmid, 1997), $T_t = \sum_{v=1}^{k} X_{t+1-v}/k$ or $T_t = c_1 med\{X_{t+1-k}, \ldots, X_t\}$. The shift from the in-control to the out-of-control state is frequently described by the change-point model satisfying $E(X_t) = \mu + aI_{\{q,q+1,\ldots\}}(t)$ with $a \in I\mathbb{R} - \{0\}$ and μ being the in-control mean of $\{X_t\}$. In case θ_t is the variance of $\{X_t\}$, possible choices would be, e.g., $T_t = (X_t - \mu)^2$ (e.g., Schipper and Schmid, 2001) or $T_t = \sum_{v=1}^{k}(X_{t+1-v} - \mu)^2/k$. The corresponding change-point model for the variance is often chosen to guarantee $\mathrm{Var}(X_t) = \Delta\sigma^2$ for $t \geq q$ and $\Delta \in (0, \infty) - \{1\}$, with σ^2 being the in-control variance of the process.

The control statistic is obtained by applying the EWMA recursion to T_t. This leads to

$$Z_t = \begin{cases} Z_0, & t = 0 \\ (1-\lambda)Z_{t-1} + \lambda T_t, & t = 1, 2, \ldots, \end{cases} \tag{1}$$

with an initial value Z_0. Z_0 is assumed to be a deterministic quantity. The parameter λ is a smoothing parameter taking values within $(0, 1]$. It equals the weight of the most recent observed value. By recursive substitution, it can be shown that Z_t is a weighted sum of the observations available at time point $t \geq 1$:

$$Z_t = \lambda \sum_{i=0}^{t-1}(1-\lambda)^i T_{t-i} + (1-\lambda)^t Z_0 \tag{2}$$

$$= \theta + \lambda \sum_{i=0}^{t-1}(1-\lambda)^i (T_{t-i} - \theta) + (1-\lambda)^t (Z_0 - \theta). \tag{3}$$