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A flexible and automated likelihood based framework for inference in stochastic volatility models

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ABSTRACT

The Laplace approximation is used to perform maximum likelihood estimation of univariate and multivariate stochastic volatility (SV) models. It is shown that the implementation of the Laplace approximation is greatly simplified by the use of a numerical technique known as automatic differentiation (AD). Several algorithms are proposed and compared with some existing maximum likelihood methods using both simulated data and actual data. It is found that the new methods match the statistical efficiency of the existing methods while significantly reducing the coding effort. Also proposed are simple methods for obtaining the filtered, smoothed and predictive values for the latent variable. The new methods are implemented using the open source software AD Model Builder, which with its latent variable module (ADMB-RE) facilitates the formulation and fitting of SV models. To illustrate the flexibility of the new algorithms, several univariate and multivariate SV models are fitted using exchange rate and equity data.

1. Introduction

Maximum likelihood (ML) estimation of nonlinear and non-Gaussian state space models has recently received a great deal of attention in the statistics literature as well as in the econometrics literature. A leading example of nonlinear and non-Gaussian state space models in financial econometrics is the class of stochastic volatility (SV) models for which the likelihood function is expressed by a high dimensional integral that cannot be evaluated analytically due to the presence of a latent volatility process. As a result, ML estimation is computationally demanding. While in the earlier literature some statistically inefficient but numerically simple methods have been proposed (e.g. Andersen and Sorensen, 1996; Harvey et al., 1994), ML remains appealing partly because of its statistical efficiency and partly because computing power is rapidly improving.

Various ML methods have been proposed in the literature. Contributions include Shephard and Pitt (1997), Durbin and Koopman (1997), Sandmann and Koopman (1998), Liesenfeld and Richard (2003, 2006), Durham (2006) and Kleppe and Skaug (2012). The idea in these papers is to evaluate the likelihood function numerically by integrating out the latent volatility process via the Laplace approximation or importance sampling techniques, followed by numerical maximization of the approximate likelihood function.

Many of the algorithms for these ML methods have been implemented in low level programming languages such as C++ or FORTRAN in order to increase the computational speed. For example, the importance sampler of Durham (2006) was implemented using FORTRAN. While these specially tailored packages are numerically efficient, the effort required to write, debug and modify the code is usually large. In addition, to use the method of Durham (2006), one has to find the analytical

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expressions for the first and second order derivatives of the joint log density of returns (observations) and volatilities (latent random variables), which are required for the Laplace approximation.

One purpose of this paper is to illustrate several algorithms for performing ML estimation of SV models using automatic differentiation (AD), combined with the Laplace approximation and importance sampling. We also present a simple empirical Bayes approach to estimation of volatilities, conditionally on the returns (observations), both as a filter and as a smoother, as well as a predictor. We then demonstrate the ease by which univariate and multivariate SV models can be explored using the latent variable module of the open source software package *AD Model Builder* (Fournier et al., 2011), referred to as *ADMB* in short. The ADMB software is available from http://admb-project.org. Historically, the latent variable module of ADMB was a separate program, but is today fully integrated with the rest of ADMB. It nevertheless has its own user manual, and we shall refer to it here as ADMB-RE. AD is a technique for calculating the exact numerical derivatives of functions defined as computer algorithms (Gallant, 1991), and should not be confused with symbolic differentiation as performed by for instance MATHEMATICA and MAPLE. As both the Laplace approximation and numerical likelihood optimization require derivatives, ADMB-RE facilitates parameter estimation and is an ideal software to do ML estimation for nonlinear and non-Gaussian state-space models.

Our paper is related to Meyer et al. (2003) where the likelihood function of the basic SV model was approximated by the Laplace approximation via AD. The focus of their paper was, however, to develop an efficient MCMC algorithm. Moreover, our method for approximating the likelihood function is different from theirs. Meyer et al. (2003) used a Kalman filter approach, where a sequence of one-dimensional Laplace approximations was used to perform the one-step updates of the Kalman filter. We apply a single multivariate Laplace approximation jointly with respect to all volatilities. While it is trivial to generalize our joint Laplace approximation to multivariate SV models, the same does not seem to be the case for the sequential univariate normal approximation of Meyer et al. Our work is also related to Liesenfeld and Richard (2006) where efficient importance sampling was used to perform both ML analysis and Bayesian analysis of SV models. The main difference between our algorithms and theirs is the different importance sampling techniques used. Finally, papers that are similar to ours in the use of a joint Laplace approximation are Durham (2006) and Martino et al. (2011), but neither of these papers use AD, and hence lack the ease and flexibility of implementation of our approach.

The rest of the paper is organized as follows. Section 2 introduces the Laplace approximation and discusses how to use it to perform classical maximum likelihood (ML) estimation. Section 3 explains how the Laplace approximation facilitates smoothing, filtering and predicting the latent variable. Section 4 describes AD and the software ADMB-RE. In Section 5 we examine the relative performance of ADMB-RE using both simulated data and actual data and demonstrate how more flexible univariate and multivariate SV models can be fitted under ADMB-RE. Section 6 concludes. Information about how to obtain the ADMB-RE code for the model developed in this paper is provided in Appendix A.

2. Maximum likelihood estimation

For illustrative purposes, we focus on the so-called basic SV model which is defined by

$$\begin{cases} X_t = \sigma_X e^{h_t/2}\epsilon_t, & t = 1, \dots, T, \\ h_{t+1} = \phi h_t + \sigma\eta_t, & t = 1, \dots, T-1, \end{cases} \tag{1}$$

where X_t is the return of an asset, $\epsilon_t \overset{\text{i.i.d.}}{\sim} N(0, 1)$, $\eta_t \overset{\text{i.i.d.}}{\sim} N(0, 1)$, $\text{corr}(\epsilon_t, \eta_t) = 0$, and $h_1 \sim N(0, \sigma^2/(1-\phi^2))$. The parameters of interest are $\theta = (\sigma_X, \phi, \sigma)'$.

Let $\mathbf{X} = (X_1, \dots, X_T)'$ and $\mathbf{h} = (h_1, \dots, h_T)'$. The likelihood function of the basic SV model is given by

$$p(\mathbf{X}; \theta) = \int p(\mathbf{X}, \mathbf{h}; \theta) d\mathbf{h} = \int p(\mathbf{X}|\mathbf{h}; \theta) p(\mathbf{h}; \theta) d\mathbf{h}. \tag{2}$$

This is a high-dimensional integral which does not have a closed form expression due to the non-linear dependence of X_t on h_t, and hence must be approximated.

Following Skaug (2002), our first algorithm (termed LA-ML) employs the Laplace approximation, that is, we match $p(\mathbf{X}, \mathbf{h}; \theta)$ and a multivariate normal distribution for $\mathbf{h}$ as closely as possible (up to a constant proportion). More precisely, the Laplace approximation to the integral (2) is

$$p(\mathbf{X}; \theta) \approx |\det(\Omega)|^{-1/2} p(\mathbf{X}, \mathbf{h}^*; \theta), \tag{3}$$

where

$$\mathbf{h}^* = \arg\max_{\mathbf{h}} \ \ln p(\mathbf{X}, \mathbf{h}; \theta) \quad \text{and} \quad \Omega = \frac{\partial^2 \ln p(\mathbf{X}, \mathbf{h}^*; \theta)}{\partial \mathbf{h} \partial \mathbf{h}'}. \tag{4}$$

The Laplace approximation is exact when $p(\mathbf{X}, \mathbf{h}; \theta)$ is Gaussian in $\mathbf{h}$. It typically works well when $p(\mathbf{h}; \theta)$ is Gaussian or nearly Gaussian and $p(\mathbf{h}; \theta)$ is more informative about $\mathbf{h}$ than $p(\mathbf{X}|\mathbf{h})$ is, in the sense of observed Fisher information. This is indeed the case for almost all the SV models used in practice. From an empirical viewpoint, the normality of $\mathbf{h}$ is documented as one of the stylized facts about volatility in Andersen et al. (2001). Theoretical results about the accuracy of

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