Computational Statistics and Data Analysis 61 (2013) 50–66

Contents lists available at SciVerse ScienceDirect

Computational Statistics and Data Analysis

journal homepage: www.elsevier.com/locate/csda

Generative models for functional data using phase and amplitude separation

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ARTICLE INFO

Article history:
Received 18 April 2012
Received in revised form 29 November 2012
Accepted 1 December 2012
Available online 10 December 2012

Keywords:
Amplitude variability
Function alignment
Function principal component analysis
Functional data analysis
Generative model
Phase variability

ABSTRACT

Constructing generative models for functional observations is an important task in statistical functional analysis. In general, functional data contains both phase (or x or horizontal) and amplitude (or y or vertical) variability. Traditional methods often ignore the phase variability and focus solely on the amplitude variation, using cross-sectional techniques such as fPCA for dimensional reduction and data modeling. Ignoring phase variability leads to a loss of structure in the data and inefficiency in data models. This paper presents an approach that relies on separating the phase (x-axis) and amplitude (y-axis), then modeling these components using joint distributions. This separation, in turn, is performed using a technique called *elastic shape analysis of curves* that involves a new mathematical representation of functional data. Then, using individual fPCAs, one each for phase and amplitude components, it imposes joint probability models on principal coefficients of these components while respecting the nonlinear geometry of the phase representation space. These ideas are demonstrated using random sampling, for models estimated from simulated and real datasets, and show their superiority over models that ignore phase-amplitude separation. Furthermore, the generative models are applied to classification of functional data and achieve high performance in applications involving SONAR signals of underwater objects, handwritten signatures, and periodic body movements recorded by smart phones.

Published by Elsevier B.V.

1. Introduction

The problem of statistical analysis and modeling of data in function spaces is important in applications arising in nearly every branch of science, including signal processing, geology, biology, and chemistry. The observations here are time samples of real-valued functions on an observation interval, and to perform effective data analysis it is desirable to have a generative, probabilistic model for these observations. The model is expected to properly and parsimoniously characterize the nature and variability in the data. It should also lead to efficient procedures for conducting hypothesis tests, performing bootstraps, and making forecasts. An interesting aspect of functional data is that underlying variability can be ascribed to two sources. In a sample data the given functions may not be perfectly aligned and the mechanism for alignment is an important topic of research. The variability exhibited in functions after alignment is termed the amplitude (or y or vertical) variability and the warping functions that are used in the alignment are said to capture the phase (or x or horizontal) variability. A more explicit mathematical definition of amplitude and phase variability will be made in Section 2. It is imperative that any technique for analysis or modeling of functional data should take both these variabilities into account.

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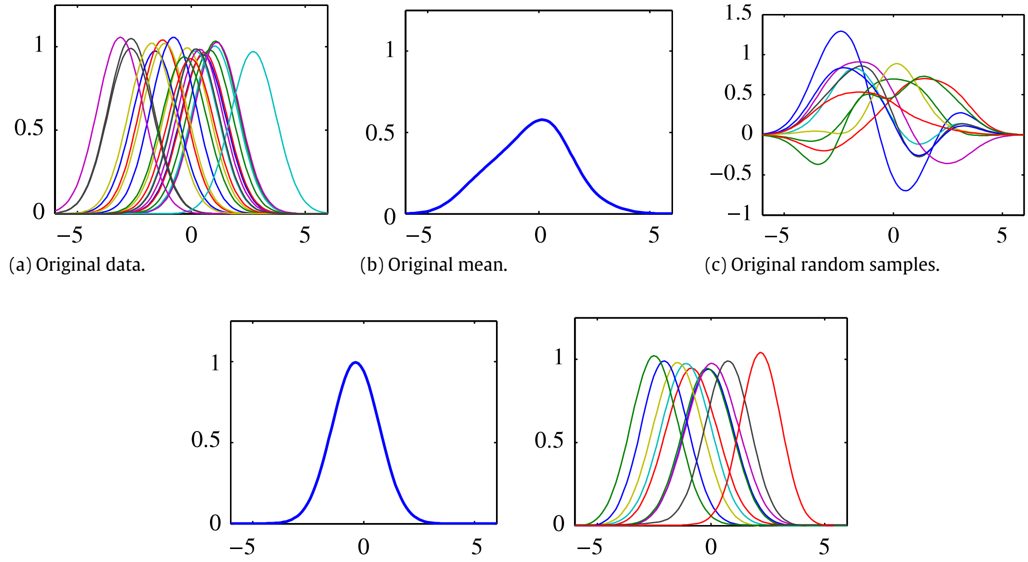
0167-9473/$ – see front matter. Published by Elsevier B.V.
doi:10.1016/j.csda.2012.12.001



(a) Original data. (b) Original mean. (c) Original random samples.

(d) Aligned mean. (e) Samples from proposed models.

Fig. 1. Samples drawn from a Gaussian model fitted to the principal components for the unaligned and aligned data.

1.1. Need for phase–amplitude separation

Many of the current methods for analyzing functional data ignore the phase variability. They implicitly assume that the observed functions are already temporally aligned and all the variability is restricted only to the y-axis. A prominent example of this situation is functional principal component analysis (fPCA) (see e.g., Ramsay and Silverman, 2005) that is used to discover dominant modes of variation in the data and has been extensively used in modeling functional observations. If the phase variability is ignored, the resulting model may fail to capture patterns present in the data and will lead to inefficient data models.

Fig. 1 provides an illustration of this using simulated functional data. This data was simulated using the equation $y_i(t) = z_i e^{-(t-a_i)^2/2}$, $t \in [-6, 6]$, $i = 1, 2, \ldots, 21$, where z_i is *i.i.d.* $\mathcal{N}(1, (0.05)^2)$ and a_i is *i.i.d.* $\mathcal{N}(0, (1.25)^2)$. The top-left plot shows the original data; each sample function here is a unimodal function with slight variability in height and a large variability in the peak placement. One can attribute different locations of the peak to the phase variability. If one takes the cross-sectional mean of this data, ignoring the phase variability, the result is shown in the top-middle plot. The unimodal structure is lost in this mean function with large amount of stretching. Furthermore, if one performs fPCA on this data and imposes the standard independent normal models on fPCA coefficients (details of this construction are given later), the resulting model will lack this unimodal structure. Shown in the top-right plot are random samples generated from such a probability model on the function space where a Gaussian model is imposed on the fPCA coefficients. These random samples are not representative of the original data; the essential shape of the function is lost, with some of the curves having two, three, or even more peaks.

The reason why the underlying unimodal pattern is not retained in the model is that the phase variability was ignored. We argue that a proper technique is to separate the phase and amplitude variability, using techniques for functional alignment, and then develop a probability model for each component. While postponing details for later, we show results obtained by a separation-based approach in the bottom row. The mean of the aligned functions is shown in the bottom left panel of Fig. 1. Clearly retained is the unimodal structure and the random samples generated under a framework that model the phase and amplitude variables individually have the same structure. Some random samples are shown in the bottom right panel; these displays are simply meant to motivate the framework and the mathematical details are provided later in the paper. This example clearly motivates the need for function alignment for modeling functional data that contains phase variability.

1.2. Past literature on phase–amplitude separation

This brings up an important question: How to separate the phase and amplitude components in a given dataset? While this is a topic of ongoing research, a number of techniques have already been discussed in the literature. The main difference between them lies in the choice of the cost function used in the alignment process. The different cost functions suggested in the statistics literature including area-under-the-curve matching (Liu and Muller, 2004; Tang and Muller,