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Adding isolated vertices makes some greedy online algorithms optimal ☆

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ABSTRACT

An unexpected difference between online and offline algorithms is observed. The natural greedy algorithms are shown to be worst case online optimal for Online Independent Set and Online Vertex Cover on graphs with "enough" isolated vertices, Freckle Graphs. For Online Dominating Set, the greedy algorithm is shown to be worst case online optimal on graphs with at least one isolated vertex. These algorithms are not online optimal in general. The online optimality results for these greedy algorithms imply optimality according to various worst case performance measures, such as the competitive ratio. It is also shown that, despite this worst case optimality, there are Freckle graphs where the greedy independent set algorithm is objectively less good than another algorithm.

It is shown that it is NP-hard to determine any of the following for a given graph: the online independence number, the online vertex cover number, and the online domination number.

1. Introduction

This paper contributes to the larger goal of better understanding the nature of online optimality, greedy algorithms, and different performance measures for online algorithms. The graph problems Online Independent Set, Online Vertex Cover and Online Dominating Set, which are defined below, are considered in the *vertex-arrival* model, where the vertices of a graph, G, are revealed one by one. When a vertex is revealed (we also say that it is "requested"), its edges to previously revealed vertices are revealed. At this point, an algorithm irrevocably either accepts the vertex or rejects it. This model is well-studied (see for example, [18,12,21,10,17,13,14]).

We show that, for some graphs, an obvious greedy algorithm for each of these problems performs less well than another online algorithm and thus is not online optimal. However, this greedy algorithm performs (at least in some sense) at least as well as any other online algorithm for these problems, as long as the graph has enough isolated vertices. Thus, in contrast to the case with offline algorithms, adding isolated vertices to a graph can improve an algorithm's performance, even making it "optimal".

For an online algorithm for these problems and a particular sequence of requests, let S denote the set of accepted vertices, which we call a *solution*. When all vertices have been revealed (requested and either accepted or rejected by the algorithm), S must fulfill certain conditions:

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- In the Online Independent Set problem [14,7], S must form an independent set. That is, no two vertices in S may have an edge between them. The goal is to maximize $|S|$.
- In the Online Vertex Cover problem [8], S must form a vertex cover. That is, each edge in G must have at least one endpoint in S. The goal is to minimize $|S|$.
- In the Online Dominating Set problem [20], S must form a dominating set. That is, each vertex in G must be in S or have a neighbor in S. The goal is to minimize $|S|$.

If a solution does not live up to the specified requirement, it is said to be infeasible. The score of a feasible solution is $|S|$. The score of an infeasible solution is ∞ for minimization problems and $-\infty$ for maximization problems. Note that for Online Dominating Set, it is not required that S form a dominating set at all times. It just needs to be a dominating set when the whole graph has been revealed. If, for example, it is known that the graph is connected, the algorithm might reject the first vertex since it is known that it will be possible to dominate this vertex later.

In Section 2, we define the greedy algorithms for the above problems, along with concepts analogous to the online chromatic number of Gyárfás et al. [11] for the above problems, giving a natural definition of optimality for online algorithms. In Section 3, we show that greedy algorithms are not in general online optimal for these problems. In Section 4, we define Freckle Graphs, which are graphs which have "enough" isolated vertices to make the greedy algorithms online optimal. In proving that the greedy algorithms are optimal on Freckle Graphs, we also show that, for Online Independent Set, one can, without loss of generality, only consider adversaries which never request a vertex adjacent to an already accepted vertex, while there are alternatives. In Section 5, we investigate what other online problems have the property that adding isolated requests make greedy algorithms optimal. In Section 6, it is shown that the online optimality results for these greedy algorithms imply optimality according to various worst case performance measures, such as the competitive ratio. In Section 7, it is shown that, despite this worst case optimality, there is a family of Freckle graphs where the greedy independent set algorithm is objectively less good than another algorithm. Various NP-hardness results concerning optimality are proven in Section 8. There are some concluding remarks and open questions in the last section. Note that Theorems 8.1 and 8.4 appeared in the second author's Master's thesis [15], which served as inspiration for this paper.

2. Algorithms and preliminaries

For each of the three problems, we define a greedy algorithm.

- In Online Independent Set, $\mathtt{GIS}$ accepts a revealed vertex, v, iff no neighbors of v have been accepted.
- In Online Vertex Cover, $\mathtt{GVC}$ accepts a revealed vertex, v, iff a neighbor of v has previously been revealed but not accepted.
- In Online Dominating Set, $\mathtt{GDS}$ accepts a revealed vertex, v, iff no neighbors of v have been accepted.

Note that the algorithms $\mathtt{GIS}$ and $\mathtt{GDS}$ are the same (they have different names to emphasize that they solve different problems). For an algorithm $\mathtt{ALG}$, we define $\overline{\mathtt{ALG}}$ to be the algorithm that simulates $\mathtt{ALG}$ and accepts exactly those vertices that $\mathtt{ALG}$ rejects. This defines a bijection between Online Independent Set and Online Vertex Cover algorithms. Note that $\mathtt{GVC} = \overline{\mathtt{GIS}}$.

For a graph, G, an ordering of the vertices, ϕ, and an algorithm, $\mathtt{ALG}$, we let $\mathtt{ALG}(\phi(G))$ denote the score of $\mathtt{ALG}$ on G when the vertices are requested in the order ϕ. We let $|G|$ denote the number of vertices in G.

For minimization problems, we define:

$$\mathtt{ALG}(G) = \max_{\phi} \mathtt{ALG}(\phi(G)).$$

That is, $\mathtt{ALG}(G)$ is the highest score $\mathtt{ALG}$ can achieve over all orderings of the vertices in G.

For maximization problems, we define:

$$\mathtt{ALG}(G) = \min_{\phi} \mathtt{ALG}(\phi(G)).$$

That is, $\mathtt{ALG}(G)$ is the lowest score $\mathtt{ALG}$ can achieve over all orderings of the vertices in G.

Since we consider a worst possible ordering, we sometimes think of an adversary as ordering the vertices.

Observation 2.1. *Let* $\mathtt{ALG}$ *be an algorithm for* Online Independent Set. *Let a graph, G, with n vertices be given. Now,* $\overline{\mathtt{ALG}}$ *is an* Online Vertex Cover *algorithm and* $\mathtt{ALG}(G) + \overline{\mathtt{ALG}}(G) = n$.

The equality $\mathtt{ALG}(G) + \overline{\mathtt{ALG}}(G) = n$ holds, since a worst ordering of G for $\mathtt{ALG}$ is also a worst ordering for $\overline{\mathtt{ALG}}$.

In considering online algorithms for coloring, [11] defines the online chromatic number, which intuitively is the best result (minimum number of colors) any online algorithm can be guaranteed to obtain for a particular graph (even when the graph, but not the ordering, is known in advance). We define analogous concepts for the problems we consider, defining for every graph a number representing the best value any online algorithm can achieve. Note that in considering all algorithms, we include those which know the graph in advance. Of course, when the graph is known, the order in which the vertices are requested is not known to an online algorithm, and the label given with a requested vertex does not necessarily correspond

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