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# Efficient enumeration of graph orientations with sources

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ABSTRACT

An orientation of an undirected graph is obtained by assigning a direction to each of its edges. It is called cyclic when a directed cycle appears, and acyclic otherwise. We study efficient algorithms for enumerating the orientations of an undirected graph. To get the full picture, we consider both the cases of acyclic and cyclic orientations, under some rules specifying which nodes are the sources (i.e. their incident edges are all directed outwards). Our enumeration algorithms use linear space and provide new bounds for the delay, which is the maximum elapsed time between the output of any two consecutively listed solutions. We obtain a delay of $O(m)$ for acyclic orientations and $\tilde{O}(m)$ for cyclic ones. When just a single source is specified, these delays become $O(m \cdot n)$ and $O(m \cdot h + h^3)$, respectively, where $h$ is the girth of the graph without the given source. When multiple sources are specified, the delays are the same as in the single source case.

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## 1. Introduction

An orientation of an undirected graph is any of the directed graphs that may be obtained by assigning directions to its edges. As such, the terminology of directed graphs applies to orientations: they are called cyclic when containing a directed cycle, and acyclic otherwise. Their sources are the nodes whose incident edges are all directed outwards. Acyclic orientations have been studied in depth. Motivated by the fact that each acyclic orientation corresponds to a partial order for the underlying graph, Iriarte [14] investigated which orientations maximize the number of linear extensions of the corresponding poset. Alon and Tarsi [1] looked for special orientations to give bounds on the size of the maximum independent set or the chromatic number. Gallai, Roy, and Vitaver independently stated that every orientation of a graph with chromatic number $k$ contains a simple directed path with $k$ nodes [11,21,29]. Johnson [16] links acyclic orientations with exactly one source to problems of network reliability.

There are further problems that can be addressed by looking at acyclic orientations. For instance, Benson et al. [5] showed that there exists a bijection between the set of the so-called superstable configurations of a graph and the set of its acyclic orientations with a unique source. Counting the number of acyclic orientations is a problem dating back to the 70 s or earlier [26], and Linial [18] proved that this problem is #P-complete. Stanley [25] showed how to count them using the chromatic polynomial (a special case of Tutte's polynomial). A related problem is the acyclic orientation game. Alon and Tuza [2] inquired about the amount of oriented edges needed to define a unique orientation, and found this number to be almost surely $\Theta(n \log n)$ in Erdős–Rényi random $n$-node graphs. Pikhurko [20] showed that the number of these edges in the worst case is no greater than $(\frac{1}{4} + o(1))n^2$ in general.

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Cyclic orientations have received less attention. Counting them is clearly #P-complete.[1] Fisher et al. [13] studied the number of dependent edges, i.e. edges generating a cycle if reversed in an orientation. This number of edges implicitly gives a hint about the number of cyclic orientations in a graph. The related problem of enumerating *strong orientations*, a special type of cyclic orientations, was also considered in [9].

We think that cyclic orientations have interesting properties to study, even though they seem more artificial and have much less applications than acyclic ones. However they can hopefully contribute to give a full picture, as acyclic and cyclic orientations correspond to an exact partition of all the $2^m$ possible orientations of a graph. As listing all orientations of a graph is straightforward, one might think that solving one problem yields the result for the other. This however is not at all satisfactory as we will see. Furthermore, techniques and invariants for one problem do not apply to the other: acyclic orientation can be obtained incrementally by keeping acyclicity as an invariant. This can be seen as a somewhat "global" property with no proper translation for the cyclic case: cyclic orientation are based on the "local" property of the existence of a directed cycle somewhere in the graph, and this cycle may appear at any time during an incremental approach. Ensuring the satisfiability of this "local" cycle-existence property, while seemingly easy, appears to yield slower algorithms than for the acyclic case. This hints at the possibility of the problem being harder, or simply that stronger techniques are required.

*Problems studied.* Our paper considers both acyclic and cyclic orientations, investigating how to enumerate them. We will use the terms enumerating and listing interchangeably. We are given an undirected connected graph $G(V, E)$ without self loops, where $n = |V|$ and $m = |E|$, and we will study the following cases.

**Single Source Orientations** (SSO): Given a node $s \in V$, enumerate all the orientations of $G$, such that $s$ is the only source.
**Single Source Acyclic Orientations** (SSAO): Given a node $s \in V$, enumerate all the acyclic orientations of $G$, such that $s$ is the only source.
**Acyclic Orientations** (AO): Enumerate all the acyclic orientations of $G$.
**Single Source Cyclic Orientations** (SSCO): Given a node $s \in V$, enumerate all the cyclic orientations of $G$, such that $s$ is the only source.
**Cyclic Orientations** (CO): Enumerate all the cyclic orientations of $G$.

As we will see the above variations of orientations provide an interesting playground for exploring enumerations algorithms on graphs. We propose new efficient algorithms and analyze their cost in terms of *delay*, which is a well-known measure of performance for enumeration algorithms corresponding to the worst-case time between any two consecutively enumerated solutions (e.g. [17]). We will focus on algorithms with guaranteed delay and space. Furthermore, we show in Section 8 how the proposed algorithms can be used to solve a wider range of orientation listing problems, in particular with respect to multiple sources.

*Previous work.* We are not aware of any listing algorithm for SSO. The result of [27] gives necessary and sufficient conditions for the existence of a single source orientation and a search algorithm to find few of them. However, this algorithm cannot be easily revised for enumeration purposes.

Problem AO has been investigated by Squire [24]. His algorithm has a good amortized cost of $O(n)$ time per solution, but the delay can be $O(n^3)$ time. The algorithm by Barbosa and Szwarcfiter [4] solves AO with an amortized time complexity of $O(n + m)$ per solution, and the delay is $O(n \cdot m)$. An alternative way of solving AO is by applying any algorithm for maximal feedback arc set enumeration (after replacing each edge with a double arc). State of the art approaches for the latter problem incur in a delay of $\Omega(n^3)$ as shown by Schwikowski and Speckenmeyer [22].

It should be noted that AO has a correspondence with cell enumeration in an arrangement of hyperplanes, which has been solved in the paradigm of reverse search [3]. Even though such algorithm can be designed to be memory efficient, e.g. by employing the techniques in [10], its delay is $\Omega(n \cdot m)$.

SSAO seems to be harder to solve, as all the techniques above for AO (including the one in [24]) do not extend smoothly. Johnson [16] presented a backtracking algorithm for SSAO, aimed at solving problems on network reliability. However its complexity is hard to estimate, as it is based on a backtracking approach with dead ends. For this reason Squire [24] writes that he has been unable to efficiently implement Johnson's approach. As far as we know, there are no provably good bounds in the literature for problems SSAO and their variants, other than the special case in which $G$ is a planar biconnected graph, and a single target $t$ adjacent to $s$ is allowed [23].

As for SSCO and CO, the best known method is to enumerate them by difference: go through all possible $2^m$ orientations and eliminate, say, the $\alpha$ acyclic ones. This method does not guarantee polynomial delay as the number $\beta = 2^m - \alpha$ of cyclic orientations can be much larger or much smaller than $\alpha$: for example, a tree with $m$ edges has $\alpha = 2^m$ and $\beta = 0$. On the other hand, a clique with $n$ nodes and $m = \frac{n \cdot (n-1)}{2}$ edges has $\alpha = n!$, i.e. the possible transitive tournaments [19], and $\beta = 2^m - n!$. As $2^m$ grows faster than $n!$, the ratio $\alpha/\beta = n!/(2^m - n!)$ tends to 0 for increasing $n$.

**Results**. For SSO and SSAO we design the first enumeration algorithms with guaranteed delay of $O(m)$ and $O(m \cdot n)$, respectively, using $O(m)$ space. For AO, we show how to obtain $O(m)$ delay: even if the latter result does not improve the amortized cost of $O(n)$ in [4,24], it improves their delay and that of [22]. For CO and SSCO, we provide the first enumeration algorithms with guaranteed delay, respectively, of $\tilde{O}(m)$ and $O(m \cdot h + h^3)$, where $h < n$ is the girth (length of the shortest

[1] For a graph with $m$ edges, there are $2^m$ orientations, which are either cyclic or acyclic: we have seen that counting the latter ones is #P-complete [18].

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