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On the complexity of rainbow coloring problems[☆]

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ABSTRACT

An edge-colored graph G is said to be *rainbow connected* if between each pair of vertices there exists a path which uses each color at most once. The *rainbow connection number*, denoted by $\mathrm{rc}(G)$, is the minimum number of colors needed to make G rainbow connected. Along with its variants, which consider vertex colorings and/or so-called strong colorings, the rainbow connection number has been studied from both the algorithmic and graph-theoretic points of view.

In this paper we present a range of new results on the computational complexity of computing the four major variants of the rainbow connection number. In particular, we prove that the STRONG RAINBOW VERTEX COLORING problem is NP-complete even on graphs of diameter 3, and also when the number of colors is restricted to 2. On the other hand, we show that if the number of colors is fixed then all of the considered problems can be solved in linear time on graphs of bounded treewidth. Moreover, we provide a linear-time algorithm which decides whether it is possible to obtain a rainbow coloring by saving a fixed number of colors from a trivial upper bound. Finally, we give a linear-time algorithm for computing the exact rainbow connection numbers for three variants of the problem on graphs of bounded vertex cover number.

1. Introduction

The concept of rainbow connectivity was introduced by Chartrand, Johns, McKeon, and Zhang in 2008 [8] as an interesting connectivity measure motivated by recent developments in the area of secure data transfer. Over the past years, this strengthened notion of connectivity has received a significant amount of attention in the research community. The applications of rainbow connectivity are discussed in detail for instance in the recent survey [25], and various bounds are also available in [10,26].

An edge-colored graph G is said to be *rainbow connected* if between each pair of vertices a, b there exists an $a - b$ path which uses each color at most once; such a path is called *rainbow*. The minimum number of colors needed to make G rainbow connected is called the *rainbow connection number* (rc), and the RAINBOW COLORING problem asks to decide if the rainbow connection number is upper-bounded by a number specified in the input. Precise definitions are given in Section 2.

The rainbow connection number and RAINBOW COLORING have been studied from both the algorithmic and graph-theoretic points of view. On one hand, the exact rainbow connection numbers are known for a variety of simple graph

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classes, such as wheel graphs [8], complete multipartite graphs [8], unit interval graphs [29], and threshold graphs [6]. On the other hand, Rainbow Coloring is a notoriously hard problem. It was shown by Chakraborty et al. [5] that already deciding if $\mathrm{rc}(G) \leq 2$ is NP-complete, and Ananth et al. [1] showed that for any $k > 2$ deciding $\mathrm{rc}(G) \leq k$ is NP-complete. In fact, Chandran and Rajendraprasad [6] strengthened this result to hold for chordal graphs. In the same paper, the authors gave a linear time algorithm for rainbow coloring split graphs which form a subclass of chordal graphs with at most one more color than the optimum. Basavaraju et al. [2] gave an $(r + 3)$-factor approximation algorithm to rainbow color a general graph of radius r. Later on, the inapproximability of the problem was investigated by Chandran and Rajendraprasad [7]. They proved that there is no polynomial time algorithm to rainbow color graphs with less than twice the minimum number of colors, unless $\mathrm{P} = \mathrm{NP}$. For chordal graphs, they gave a $5/2$-factor approximation algorithm, and proved that it is impossible to do better than $5/4$, unless $\mathrm{P} = \mathrm{NP}$.

Several variants of the notion of rainbow connectivity have also been considered. Indeed, a similar concept was introduced for vertex-colored graphs by Krivelevich and Yuster [22]. A vertex-colored graph H is *rainbow vertex connected* if there is a path whose internal vertices have distinct colors between every pair of vertices, and this gives rise to the *rainbow vertex connection number* (rvc). The *strong rainbow connection number* (src) was introduced and investigated also by Chartrand et al. [10]; an edge-colored graph G is said to be *strong rainbow connected* if between each pair of vertices a, b there exists a shortest $a - b$ path which is rainbow. The combination of these two notions, *strong rainbow vertex connectivity* (srvc), was studied in a graph theoretic setting by Li et al. [24].

Not surprisingly, the problems arising from the strong and vertex variants of rainbow connectivity are also hard. Chartrand et al. showed that $\mathrm{rc}(G) = 2$ if and only if $\mathrm{src}(G) = 2$ [8], and hence deciding if $\mathrm{src}(G) \leq k$ is NP-complete for $k = 2$. The problem remains NP-complete for $k > 2$ for bipartite graphs [1], and also for split graphs [21]. Furthermore, the strong rainbow connection number of an n-vertex bipartite graph cannot be approximated within a factor of $n^{1/2-\epsilon}$, where $\epsilon > 0$ unless $\mathrm{NP} = \mathrm{ZPP}$ [1], and the same holds for split graphs [21]. The computational aspects of the rainbow vertex connection numbers have received less attention in the literature. Through the work of Chen et al. [12] and Chen et al. [11], it is known that deciding if $\mathrm{rvc}(G) \leq k$ is NP-complete for every $k \geq 2$. However, to the best of our knowledge, the complexity of deciding whether $\mathrm{srvc}(G) \leq k$ (the k-SRVC problem) has not been previously considered.

In this paper, we present new positive and negative results for all four variants of the rainbow coloring problems discussed above.

- In Section 3, we prove that k-SRVC is NP-complete for every $k \geq 3$ even on graphs of diameter 3. Our reduction relies on an intermediate step which proves the NP-hardness of a more general problem, the k-Subset Strong Rainbow Vertex Coloring problem. We also provide bounds for approximation algorithms (under established complexity assumptions), see Corollary 6, and tighten the hardness result to additionally cover 2-SRVC.
- In Section 4, we show that all of the considered problems can be formulated in monadic second order (MSO) logic. In particular, this implies that for every fixed k, all of the considered problems can be solved in linear time on graphs of bounded treewidth, and the vertex variants can be solved in cubic time on graphs of bounded clique-width.
- In Section 5, we investigate the problem from a different perspective: we ask whether, given an n-vertex graph G and an integer k, it is possible to color G using k colors less than the known upper bound. Here we employ a win–win approach and show that this problem can be solved in time $\mathcal{O}(n)$ for any fixed k.
- In Section 6, we show that in the general case when k is not fixed, three of the considered problems admit linear-time algorithms on graphs of bounded vertex cover number. This is also achieved by exploiting a win–win approach, where we show that either k is bounded by a function of the vertex cover number and hence we can apply the result of Section 4, or k is sufficiently large which allows us to exploit the structure of the graph and solve the problem directly.

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2. Preliminaries

2.1. Graphs and rainbow connectivity

We refer to [15] for standard graph-theoretic notions. We use $[i]$ to denote the set $\{1, 2, \ldots, i\}$. All graphs considered in this paper are simple and undirected. The *degree* of a vertex is the number of its incident edges, and a vertex is a *pendant* if it has degree 1. We will often use the shorthand ab for the edge $\{a, b\}$. For a vertex set X, we use $G[X]$ to denote the subgraph of G induced on X.

A *vertex coloring* of a graph $G = (V, E)$ is a mapping from V to $\mathbb{N}$, and similarly an *edge coloring* of G is a mapping from E to $\mathbb{N}$; in this context, we will often refer to the elements of $\mathbb{N}$ as colors. An *$a - b$ path P of length p* is a finite sequence of the form $(a = v_0, e_0, v_1, e_1, \ldots, b = v_p)$, where $v_0, v_1, \ldots, v_p$ are distinct vertices and $e_0, \ldots, e_{p-1}$ are distinct edges and each edge e_j is incident to v_j and v_{j+1}. An $a - b$ path of length p is a *shortest path* if every $a - b$ path has length at least p. The *diameter* of a graph G is the length of its longest shortest path, denoted by $\mathrm{diam}(G)$. Given an edge (vertex) coloring α of G, a color $x \in \mathbb{N}$ *occurs* on a path P if there exists an edge (an internal vertex) z on P such that $\alpha(z) = x$.

A vertex or edge coloring of G is *rainbow* if between each pair of vertices a, b there exists an $a - b$ path P such that each color occurs at most once on P; in this case we say that G is *rainbow connected* or *rainbow colored*. We denote by $\mathrm{rc}(G)$ the