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Computational complexity of distance edge labeling[☆,☆☆]

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ABSTRACT

The problem of Distance Edge Labeling is a variant of Distance Vertex Labeling (also known as $L_{2,1}$ labeling) that has been studied for more than twenty years and has many applications, such as frequency assignment.

The Distance Edge Labeling problem asks whether the edges of a given graph can be labeled such that the labels of adjacent edges differ by at least two and the labels of edges at distance two differ by at least one. Labels are chosen from the set $\{0, 1, \ldots, \lambda\}$ for λ fixed.

We present a full classification of its computational complexity—a dichotomy between the polynomial-time solvable cases and the remaining cases which are NP-complete. We characterize graphs with $\lambda \leq 4$ which leads to a polynomial-time algorithm recognizing the class and we show NP-completeness for $\lambda \geq 5$ by several reductions from Monotone Not All Equal 3-SAT.

Moreover, there is an absolute constant $c > 0$ such that there is no 2^{cn}-time algorithm deciding the Distance Edge Labeling problem unless the exponential time hypothesis fails.

1. Introduction

We study the computational complexity of the Distance Edge Labeling problem. This result extend a result presented at the IWOCA 2015 conference [23]. This problem belongs to a wider class of problems that generalize the graph coloring problem. The task is to assign a set of colors to each vertex, such that whenever two vertices are adjacent, their colors differ from each other. For a survey about this famous graph problem and related algorithms, see a survey by Formanowicz and Tanaś [13].

We are interested in the so-called *distance labeling*. In this generalization of the former problem the condition enforcing different colors is extended and takes into account also the second neighborhood of a vertex (or an edge). The second neighborhood is the set of vertices (or edges) at distance at most 2. For a survey about distance labelings, we refer to the article by Tiziana Calamoneri [6], as well as her updated online survey [7].

Graph distance labeling has been first studied by Griggs and Yeh [15,30] in 1992. The problem has many applications, the most important one being frequency (channel) assignment [1,16]. For an extensive list of applications read a survey about

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application of generalized colorings by Roberts [26]. The complexity of the Distance Vertex Labeling problem has been established by Fiala et al. [11]. They showed a dichotomy between polynomial cases for $\lambda \leq 3$ and NP-complete cases for $\lambda \geq 4$.

A related graph coloring, the (p, q)-total labeling [21], is also studied. In this labeling the task is to label vertices and edges in such a way that vertices joined by an edge and edges sharing a vertex receive labels at least q apart, while a vertex and an incident edge receive labels at least p apart. The complexity of finding an admissible (p, q)-total labeling when restricted to tree only has been determined by Hasunuma et al. [18]. For general graphs this question is solved only partially [20] in the special case of the $(p, 1)$-total labeling.

Moreover, for the usual graph coloring problem there is a theorem of Vizing [29], which states that for the chromatic index $\chi'(G)$ it holds that $\Delta \leq \chi' \leq \Delta + 1$, where Δ is the maximum degree of the graph. For $\mathrm{L}_{2,1}$ labeling there is a general bound due to Havet et al. [19], namely $\lambda \leq \Delta^2$, for $\Delta \geq 10^{69}$. This partially answers on the general conjecture ($\lambda \leq \Delta^2$ for any graph) posed by Griggs and Yeh [15].

Before we proceed to the formal definition of the corresponding decision problem, we give several definitions of a labeling mapping of a graph and of the minimal distance edge labeling number. Note that the distance edge labeling is equivalent to the distance vertex-labeling of the associated line-graphs. A *line-graph* $L(G)$ is a graph derived from another graph G such that vertices of $L(G)$ are edges of G and two vertices a, b of $L(G)$ are connected by an edge whenever a, b (as edges of G) are adjacent. We define the *distance* between edges of a graph as their distance in the corresponding line-graph.

Definition 1.1 (*Edge Labeling Mapping*). Let $G = (V, E)$ be a graph. A mapping $f'_{2,1}: E \to \mathbb{N}$ is an *edge labeling*, if it satisfies:

- $|f'_{2,1}(e) - f'_{2,1}(e')| \geq 2$ for neighboring edges e and e' (i.e. those at distance 1),
- $|f'_{2,1}(e) - f'_{2,1}(e')| \geq 1$ for edges e and e' at distance 2.

As usual, we are interested in a labeling that minimizes the greatest label used by a feasible labeling.

Definition 1.2 (*Minimum Distance Edge Labeling*). Let $G = (V, E)$ be a graph and $f'_{2,1}$ an edge labeling mapping, we define the graph parameter $\lambda'_{2,1}$ as:

$$\lambda'_{2,1}(G) := \min_{f'_{2,1}} \max_{e \in E} f'_{2,1}(e).$$

The size of the range of a (not necessarily optimal) edge labeling mapping $f'_{2,1}$ is called the *span*.

Definition 1.3 (Distance Edge Labeling *Problem (Also Known As* $\mathrm{L}'_{2,1}$*)*).

Input:	A graph G.
Parameter:	$\lambda \in \mathbb{N}$.
Question:	Is $\lambda'_{2,1}(G) \leq \lambda$?

1.1. Our results

Our main result is the following theorem about the dichotomy of the Distance Edge Labeling problem.

Theorem 1.4 (*Dichotomy of Distance Edge Labeling*). *The* Distance Edge Labeling *problem is polynomial-time solvable if and only if* $\lambda \leq 4$. *Otherwise it is* NP*-complete.*

We derive Theorem 1.4 as a combination of Theorem 3.1 that describes all graphs with $\lambda'_{2,1} \leq 4$ and Theorem 4.1 presenting the NP-completeness result. Note that Theorem 4.1 also extends to the following inapproximability result:

Corollary 1.5. *The* Distance Edge Labeling *problem cannot be approximated within a factor of* $6/5 - \varepsilon$, *unless* P = NP.

Moreover, according to [12], the proof implies that the Distance Edge Labeling is paraNP-hard while parameterized by its span.

Using the well-established exponential time hypothesis it is possible to prove an exponential lower bound for a fixed span greater than 5.

Corollary 1.6 (*An Exponential Lower Bound for Distance Edge Labeling*). *For every fixed span* λ *fulfilling* $\lambda \geq 5$ *there is a positive real* s *such that the* Distance Edge Labeling *problem parameterized by its size* n *cannot be solved in time* $2^{sn}n^{O(1)}$, *unless* ETH *fails.*

Organization of the paper. We give some immediate observation in Section 2. The proof of Theorem 1.4 is split into two parts—the algorithmic and the hardness parts. Section 3 is devoted to the algorithmic part of the proof, while the hardness proof is presented in Section 4. In Section 5 we give a brief description of the ETH and derive an algorithmic lower bound, based on this conjecture, for the Distance Edge Labeling problem. We enclose the paper with some open problems.