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Grid spanners with low forwarding index for energy efficient networks[☆]

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ABSTRACT

A routing R of a connected graph G is a collection that contains simple paths connecting every ordered pair of vertices in G. The *edge-forwarding index with respect to* R (or simply the forwarding index with respect to R) $\pi(G, R)$ of G is the maximum number of paths in R passing through any edge of G. The *forwarding index* $\pi(G)$ of G is the minimum $\pi(G, R)$ over all routings R's of G. This parameter has been studied for different graph classes (Xu and Xu, 2012; Bouabdallah and Sotteau, 1993; Fernandez de la Vega and Gordone, 1992; de la Vega and Manoussakis, 1992). Motivated by energy efficiency, we look, for different numbers of edges, at the best spanning graphs of a square grid, namely those with a low forwarding index.

1. Introduction

A routing R of a given connected graph G of order N is a collection of $N(N-1)$ simple paths connecting every ordered pair of vertices of G. The routing R induces on every edge e a *load* that is the number of paths going through e. The *edge-forwarding index* (or simply the forwarding index) $\pi(G, R)$ of G with respect to R is the maximum number of paths in R passing through any edge of G. It corresponds to the maximum load over all edges of the graph when R is used. Therefore, it is important to find routings minimizing this index. The forwarding index $\pi(G)$ of G is the minimum $\pi(G, R)$ over all routings R's of G.

The forwarding-index was introduced by Chung & Al in 1987 [9]. Due to its importance, this parameter has been studied quite extensively: on one side results have been given for different graph classes (e.g. random graphs [12], transitive and Cayley graphs [21,31], graphs with small numbers of vertices [3] and well-connected graphs [10]). On the other side deep relations with other expansion-related graph invariants have been established: Laplacian, Cheeger constant (see the survey [25]), Sparsest cut [22] and the "geometry of graphs" [23]. This notion has also been used to prove that some Markov chains mix fast using either canonical paths (routings) or "resistance" [30]. See the recent survey [35] for a global view on the known results.

We call a connected spanning subgraph of a graph G, a *spanner* of G. More precisely, it is a connected subgraph that has the same set of vertices as G. Our goal is to find, for a given bound on the number of edges, the best spanner of G, namely the one with the minimum forwarding index. The problem can also be viewed as: for a given bound U on the forwarding index, find a spanner F of G with minimum number of edges such that $\pi(F) \leq U$.

Knowing how to solve this problem is very interesting in practice for network operators willing to reduce the energy consumed by their networks. In fact, most of the network links consume a constant energy independently of the amount of

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traffic they are flowing [5], [26]. Therefore, it was proposed to reduce the energy used by the network links by turning some of them off, or more conveniently, putting them into an idle mode. Outside the rush hours, several studies [2,8,16,17], show that a good choice of the links to turn off can lead to significant energy savings, while keeping the same communication quality. In the case where the flows from every node to every other node are of the same order, and where the capacities also lie in the same small range, a good choice of those links is reduced to the problem of finding spanners of the network with low forwarding indices.

In this paper, we consider the case in which the initial graph is a *square grid*. Backbone networks are generally not modeled as a grid, as showed with the typical models found in SNDLib (http://sndlib.zib.de/) and studied, e.g. in [15]. However, a large number of networks are modeled by a grid in the literature. We may cite: wireless network [13], such as, wireless adhoc sensor networks [24], or random wireless networks [34], RFID reader antenna network [27,33], mobile ad hoc networks [28], urban mesh access networks [4], femto cell networks [6], wireless backhaul networks [7], cellular networks [1], interconnection networks [32], optically interconnected arrays [19], stochastic geometry and random graphs [20]. More importantly, we wanted to understand well the difficulty of the problem on simple graphs. We thus choose to study the square grids, as they are a classical family of graphs. They are also a simple case of planar graphs. Solving the problem for square grids give hints to solve the more general case of planar graphs with bounded degrees, as they can be embedded in a grid [29]. So the case of the grid is to be considered as a paradigm or a *typical planar* graph rather than an actual example of an existing network.

We consider the asymptotic case with n large. We have two main contributions.

On one side, it is well-known that the forwarding index of the $n \times n$ grid G_n is $\frac{n^3}{2}$ (see Proposition 1 [14]). An important remark is, that the load of the associated routing on the $2(n-1)^2 \sim 2n^2$ edges is lower in the corner than in the middle of the grid. Using this fact, we show how to build spanners of G_n with much fewer edges (only $13/18 \approx 72\%$ of the edges) and the same forwarding indices as G_n. We then demonstrate that our spanners are close to optimum, in the sense that we prove that it is impossible to build spanners with fewer than $4/3n^2$ edges (66% of the edges).

On the other side, the smallest possible spanner of the $n \times n$ grid G_n is a spanning tree. The forwarding index of the best spanning tree is asymptotically $\frac{3n^4}{8}$, see Proposition 2 [14]. When we consider spanners with a larger number of edges, the load on the edges decreases, and so does the forwarding index. In this paper, we study how the forwarding index decreases, when we increase the number of edges. The following table summarizes our results. One interesting fact is that, with $n^2 + a^2$ edges (i.e. a^2 extra edges), the forwarding index has order $\Theta(\frac{n^4}{a})$. This is due to the planarity of the grid.

	Spanning tree	Spanners For an integer a, $2 \le a \le n$		Grid
Forwarding index	$\frac{3}{8}n^4$	$\frac{1}{2a}n^4$	$\frac{1}{2}n^3$	$\frac{1}{2}n^3$
Lower bound on number of edges	n^2-1	$\simeq n^2 + \frac{4}{9}(0.1a)^2$	$\frac{12}{9}n^2$	
Number of edges in constructions	n^2-1	$n^2 + \frac{4}{9}a^2$	$\frac{13}{9}n^2$	$2n^2$

Proposition 1 ([14]). *The forwarding index of G_n is asymptotically $\frac{n^3}{2}$.*

Proposition 2 ([14]). *For $n \ge 3$, the spanning tree of G_n with the minimum forwarding index is a tree with centroid of degree 4 and 4 branches of almost equal sizes. Its forwarding index is asymptotically $\frac{3n^4}{8}$.*

2. Spanners with the forwarding index of the grid, $\frac{n^3}{2}$, but much fewer edges

In this section, we first show that a spanner with the forwarding index of the grid has at least $\frac{4n^2}{3} = \frac{12n^2}{9}$ edges. We then provide spanners with $\frac{13n^2}{9}$ edges. But, before, we present some notations that will be used throughout the paper.

Notations. We note by $G_n = (V_n, E_n)$ the $n \times n$ square grid, where V_n is the set of vertices and E_n is the set of edges. A square grid can always be seen as n rows intersecting n columns. We name $v(r, c)$ the vertex at the intersection of row $r \in [n]$ with column $c \in [n]$, where $[n]$ denotes the interval of the integer numbers between 1 and n. An edge joining $v(r, c)$ to $v(r, c+1)$ is named $e_h(r, c)$ and an edge joining $v(r, c)$ to $v(r+1, c)$ is named $e_v(r, c)$.

Proposition 3. *For any F spanner of G_n such that $\pi(F) \le \frac{n^3}{2}$, F must have, asymptotically, at least $\frac{4n^2}{3}$ edges.*

Proof. Consider F a spanner of G_n and let R be a routing of F such that $\pi(F, R) \le \frac{n^3}{2}$. For an integer $l \in [n]$, we call *load on line l*, the sum of the load on the edges $e_v(l, j) \in E(F)$, for $j \in [n]$. The load on line l is $2l(n-l)n^2$ as there are ln vertices over line l and $(n-l)n$ vertices below. If F has $n - x_l$ edges on line l, there exists at least one of these edges with load at least $\frac{2l(n-l)n^2}{n-x_l}$. As $\pi(F, R) \le \frac{n^3}{2}$, we should have

$$\frac{2l(n-l)n^2}{n-x_l} \le \frac{n^3}{2}.$$