ARTICLE IN PRESS

Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The caterpillar-packing polytope

Javier Marenco *

Sciences Institute, National University of General Sarmiento, Argentina
Computer Science Department, FCEyN, University of Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 23 December 2015
Received in revised form 8 February 2017
Accepted 30 March 2017
Available online xxxx

Keywords:
Caterpillar-packing
Facets

ABSTRACT

A *caterpillar* is a connected graph such that the removal of all its vertices with degree 1 results in a path. Given a graph G, a *caterpillar-packing of* G is a set of vertex-disjoint (not necessarily induced) subgraphs of G such that each subgraph is a caterpillar. In this work we consider the set of caterpillar-packings of a graph, which corresponds to feasible solutions of the *2-schemes strip cutting problem with a sequencing constraint* (2-SSCPsc) presented by F. Rinaldi and A. Franz in 2007. We study the polytope associated with a natural integer programming formulation of this problem. We explore basic properties of this polytope, including a lifting lemma and several facet-preserving operations on the graph. These results allow us to introduce several families of facet-inducing inequalities.

1. Introduction

A *caterpillar* is a connected graph such that the removal of all its vertices with degree 1 results in a path, see Fig. 1(a). Given a graph $G = (V, E)$, the subgraph induced by an edge subset $C \subseteq E$ is $G_C = (V_C, C)$, where $V_C \subseteq V$ is the set of endpoints of the edges in C. A *caterpillar-packing of* G is a set $\mathcal{C} = \{C_1, \dots, C_k\}$ such that (a) $C_i \subseteq E$ and G_{C_i} is a caterpillar, for $i = 1, \dots, k$, and (b) $V_{C_i} \cap V_{C_j} = \emptyset$, for $i \neq j$. In other words, a caterpillar-packing of G is a set of vertex-disjoint (not necessarily induced) subgraphs of G such that each subgraph is a caterpillar.

The interest in caterpillar-packings comes from an integer programming approach to the *2-schemes strip cutting problem with a sequencing constraint* (2-SSCPsc), a problem that arises in the context of corrugated cardboard machines. The 2-SSCPsc was first presented in [13], where it was shown to be $\mathcal{NP}$-hard. In this problem, a set $\mathcal{O}$ of orders must be scheduled for production, and the corrugating machine can manufacture up to two orders at the same time. Any feasible combination of two orders to be produced is called a *2-scheme*, whereas any feasible way of producing one order is called a *1-scheme* in this context. Since the orders may require different quantities to be produced, more than one scheme may be necessary to produce a given order, hence a standard cutting stock approach is employed to solve the 2-SSCPsc. However, the 2-SSCPsc asks for an additional constraint: every order must appear in consecutive schemes. This implies that the solution is no longer a set of schemes but a sequence of schemes instead, such that all the orders are produced up to the required quantities and such that every order appears in consecutive schemes. This last constraint makes the problem quite difficult in practice.

If we define the *schemes graph* to be $SG = (\mathcal{O}, S)$, where $\mathcal{O}$ is the set of orders and $S = \{ij$: there exists a feasible 2-scheme with orders i and $j\}$, then the 2-schemes present in any feasible solution induce a caterpillar-packing in SG [13]. This observation motivated the introduction in [10] of an integer programming model aiming to exploit this structure. Although the computational results reported in this work were quite reasonable, optimality is not always achieved in large real-life instances, thus suggesting that a comprehensive exploration of the associated polytope may be necessary in order to make progress at solving the 2-SSCPsc. In this work we start this issue.

* Correspondence to: Computer Science Department, FCEyN, University of Buenos Aires, Argentina.
E-mail address: jmarenco@ungs.edu.ar.

http://dx.doi.org/10.1016/j.dam.2017.03.018
0166-218X/

Please cite this article in press as: J. Marenco, The caterpillar-packing polytope, Discrete Applied Mathematics (2017), http://dx.doi.org/10.1016/j.dam.2017.03.018

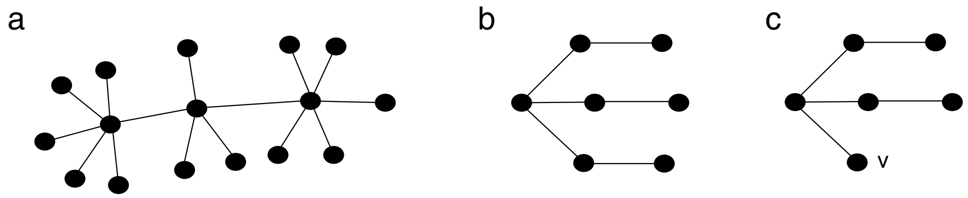

Fig. 1. (a) A caterpillar. (b) The bipartite claw. (c) The incomplete bipartite claw, with v its dangling vertex.

If $C \subseteq E$ is a set of edges of G, denote by $\chi^C \in \{0, 1\}^{|E|}$ its characteristic vector, i.e., $\chi^C_e = 1$ if and only if $e \in C$, for every $e \in E$. We define the *caterpillar-packing polytope* associated to the graph G to be

$$CPP(G) = \text{conv}\{\chi^C : C \text{ is the edge set of a caterpillar-packing of } G\}.$$

Families of valid inequalities for $CPP(G)$ may be incorporated to a cutting-plane-based procedure for the 2-SSCPsc, thus motivating the study of this polytope.

Caterpillar-based structures in graphs have been tackled with integer programming techniques in previous works. The *minimum spanning caterpillar problem* asks for a spanning caterpillar minimizing a linear cost function that assigns different costs to leaf edges and edges from the central path. Integer programming and heuristic approaches for this problem are presented in [14,15]. Theoretical developments concerning its approximability include [5,6]. This problem is related to the *minimum ring-star problem*, where the central path of the spanning caterpillar is replaced by a cycle. Both integer programming [7,8] and heuristic [3,4] approaches have been pursued for this problem. The natural generalization of this problem asking for more than one cycle is called the *m-ring-star problem* and has been studied in [1,12,16,17].

This paper is organized as follows. Section 2 presents a natural integer programming formulation whose associated polytope is $CPP(G)$ and provides some straightforward properties of this polytope. In particular, facetness results for the model constraints and a lifting lemma are presented, and these results are useful in the following sections. This polytope admits facet-preserving procedures, which take as input a facet-inducing inequality and generate a slightly larger valid inequality that, under additional hypotheses, also induces a facet. Section 3 introduces these procedures and proves them correct. Section 4 presents preliminary computational experiments with these procedures. Finally, Section 5 states some conclusions and open problems. The results contained in this work appeared without proof in the extended abstract [11].

2. Formulation and basic properties

We introduce in this section a natural integer programming formulation for $CPP(G)$, based on the following classical result. The *bipartite claw* is the graph depicted in Fig. 1(b).

Theorem 1 ([9]). *A connected graph H is a caterpillar if and only if H does not contain any cycle and any bipartite claw.*

We assume $E \neq \emptyset$ throughout this work. We introduce a binary variable x_e for each $e \in E$, such that $x_e = 1$ if and only if the solution includes the edge e. We denote by $\mathcal{C}(G)$ the set of all (not necessarily induced) cycles in G, and by $\mathcal{B}(G)$ the set of all (not necessarily induced) bipartite claws of G, in both cases regarded as sets of edges. In this setting, $CPP(G)$ is the convex hull of the points $x \in \{0, 1\}^{|E|}$ satisfying the following constraints:

$$\sum_{e \in C} x_e \leq |C| - 1 \quad \forall\, C \in \mathcal{C}(G), \tag{1}$$

$$\sum_{e \in B} x_e \leq 5 \quad \forall\, B \in \mathcal{B}(G). \tag{2}$$

The *cycle constraints* (1) ask feasible solutions not to contain any cycle, whereas the *bipartite claw constraints* (2) forbid bipartite claws. Hence, Theorem 1 ensures that integer points in $CPP(G)$ represent caterpillar-packings of G. Note that the definition of a caterpillar-packing does not ask every vertex to be included in some caterpillar (e.g., the empty set of edges is a caterpillar-packing), so we do not have constraints asking for such conditions.

The set of caterpillar-packings of G is an independence system [2], thus implying some of the following straightforward properties. We define $\mathbf{u}_e \in \mathbb{R}^{|E|}$ to be the unit vector associated with the edge e, and $\mathbf{0} \in \mathbb{R}^{|E|}$ to be the all-zeros vector with $|E|$ entries.

Proposition 1. (i) *The polytope $CPP(G)$ is full-dimensional.*
(ii) *The (valid) inequality $x_e \geq 0$ is facet-inducing, for any $e \in E$.*
(iii) *The (valid) inequality $x_e \leq 1$ is facet-inducing, for any $e \in E$.*
(iv) *If $\pi x \leq \pi_0$ is any facet-inducing inequality different from $x_e \geq 0$ for any $e \in E$, then $\pi \geq 0$.*