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A min–max relation in flowgraphs and some applications

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ABSTRACT

A flowgraph $G = (V, A, s)$ is a digraph such that there is a dipath from s to u for every vertex u in G. A vertex w dominates a vertex u if and only if all dipaths in G from s to u pass through w. The vertex s dominates trivially any vertex in G. In this work, we define two sets called dominator cover and junction partition. A dominator cover in G is a set of vertices that includes s and non-trivial dominators for all vertices in G. A junction partition $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$ is a partition of V which satisfies the following property: if vertex u belongs to $\mathcal{B}_i$ and vertex v belongs to $\mathcal{B}_j$ $(i \neq j)$, then there are internally vertex-disjoint dipaths from s to u and from s to v, for short, s is a junction of u and v. The first part of this work shows that, in flowgraphs, the minimum size of a dominator cover is equal to the maximum size of a junction partition. The second part describes some applications for this relation, such as, a new correctness proof of an algorithm that finds a maximum junction partition in reducible flowgraphs; and good time and space complexity algorithms for problems related to junctions and nearest common ancestors in acyclic digraphs.

1. Introduction

This paper is the full version of the extended abstract published and presented in VIII Latin-American Algorithms, Graphs, and Optimization Symposium (LAGOS'15) [11].

In [10] is described an efficient algorithm that receives an acyclic digraph D and a vertex s of D, and returns a partition $\mathcal{B}$ of the set of vertices of D such that, taking any pair of vertices $\{u, v\}$, u and v are in different parts of $\mathcal{B}$ if and only if there are vertex-disjoint dipaths from s to u and from s to v, for short, s is a *junction* of u and v. In that paper we observed a min–max relation in *reducible flowgraphs*. Now, we generalize this relation for any flowgraph.

Frank and Gyárfás [13], Ramachandran [20], Chen and Zang [5], and Xiao [25] have studied min–max relations in reducible flowgraphs. Let G be such a graph. A collection of cycles $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ of G is called a *vertex-disjoint cycle packing* of G if each vertex v in G appears at most once in $\mathcal{C}$. A *feedback vertex set* of G is a set of vertices S such that every cycle in G has a vertex in S. Frank and Gyárfás showed that the maximum cardinality of a vertex-disjoint cycle packing of G is equal to the minimum cardinality of a feedback vertex set of G. They conjectured that the maximum cardinality of an *arc-disjoint cycle packing* of G is also equal to the minimum cardinality of a *feedback arc set* of G. Ramachandran, based on her algorithm for finding a minimum weighted feedback arc set [19], showed the Frank and Gyárfás' conjecture.

Chen and Zang observed that the Ramachandran's result could be adapted to the case where the reducible flowgraph is weighted. In a weighted reducible flowgraph, each arc e (vertex v) is associated to a non-negative integer $w(e)$ $(w(v))$. Given a weighted reducible flowgraph G, the collection $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ is a *weighted cycle packing* if each arc e (vertex v) appears at most $w(e)$ $(w(v))$ times in cycles of $\mathcal{C}$. The weight of a feedback arc set is given by the sum of weights of all its arcs.

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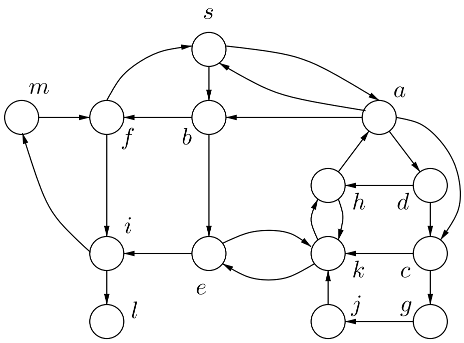

Fig. 1. A flowgraph rooted in s.

Chen and Zang [5] showed that, in weighted reducible flowgraphs, the maximum cardinality of a weighted cycle packing (as just defined) is equal to the minimum weighted feedback arc set.

Still considering weighted reducible flowgraphs, a *weighted feedback arc set packing* is defined in a similar way to the definition of weighted cycle packing. A weighted feedback arc set packing is a collection of feedback arc sets $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ such that each arc e can appear at most $w(e)$ times in feedback arc sets of $\mathcal{F}$. The weight of a cycle is defined as the sum of the weights of its arcs. Xiao [25] showed that, in a weighted reducible flowgraph, the maximum cardinality of a weighted feedback arc set packing is equal to the minimum weight of cycles.

We will describe a min–max relation that involves structures in flowgraphs named as *dominator cover* and *junction partition*. As we will see, find out relations among structures in these graphs could be interesting to solve some problems efficiently, such as, finding a representative (or all) *nearest common ancestor(s)* in acyclic digraphs, and finding a representative (or all) *junction(s)* in acyclic digraphs.

A *flowgraph* $G = (V, A, s)$ is a digraph where the vertex s reaches any vertex in G, i.e., there is in G a dipath from s to any other vertex in G. The vertex s is called the *root* of G. An example of a flowgraph is given in Fig. 1.

We always assume that $|V| = n$ and $|A| = m$. A vertex w *dominates* a vertex u if and only if all dipaths from s to u pass through w. We say that vertex s *trivially dominates* all vertices of G. Any vertex different from s which dominates u is called a *non-trivial dominator* of u. If there are at least two vertex-disjoint dipaths from s to u, then the only non-trivial dominator of u is vertex u itself.

Given a flowgraph $G = (V, A, s)$, we define a *dominator cover* $\mathcal{A}$ in G as a set that contains s, and at least one non-trivial dominator for each vertex u (different from s) in G. Formally, a dominator cover in G is a set $\mathcal{A} = \cup_{u \in G} \mathcal{A}_u$ where $\mathcal{A}_s = \{s\}$, and $\mathcal{A}_u$ is a non-empty set of non-trivial dominators of u for all u different from s in G. Note that we could have more than one dominator cover in G. For example, a possible dominator cover in the flowgraph in Fig. 1 is the set $\mathcal{A}^1 = \{s, a, b, c, e, f, g, h, i, k\}$.

To see why $\mathcal{A}^1$ is a dominator cover, observe that vertices s, a, b, c, e, f, h, i and k have in-degree more than 1, and they are non-trivial dominators from themselves (except s which is in the set by definition). The vertices a, c, g, and i are parents of vertices whose in-degree is equal to 1, so they are non-trivial dominators of the remaining vertices. However, the smaller set $\mathcal{A}^2 = \{s, a, b, e, f, h, i, k\}$ is also a dominator cover: a dominates (non-trivially) a, c, d, g and j; i dominates i, l and m; and the remaining vertices in $\mathcal{A}^2$ dominate themselves. Our first interest is *to find out a cover with the minimum number of dominators.*

Given a flowgraph $G = (V, A, s)$, the set $\mathcal{B} = \{\mathcal{B}_1, \ldots, \mathcal{B}_k\}$ is a *junction partition* of G if it is a partition of V, and the following property holds:

> if vertex u belongs to $\mathcal{B}_i$ and vertex v belongs to $\mathcal{B}_j$ $(i \neq j)$, then s is a junction of vertices u and v.

The *size* of a junction partition is the number of subsets belonging to it.

The partition $\{\{s\}, \{a, c, d, g, h, j, k\}, \{b, e, f, i, l, m\}\}$ is a junction partition of size 3 of the flowgraph in Fig. 1. However, the partition given next is a junction partition of size 4: $\{\{s\}, \{a, c, d, g, h, j, k\}, \{b, e, f\}, \{i, l, m\}\}$. It is not so hard to check that these partitions are both junction partitions of the flowgraph considered. Another interest of this work is *to find out a junction partition with maximum size.*

Considering yet a flowgraph $G = (V, A, s)$, we describe next important concepts for this work such as *immediate dominator*, *farthest dominator*, and *dominator tree*. For any vertex $u \neq s$, the *immediate dominator* of u is the unique vertex $w \neq u$ that dominates u and is dominated by all dominators of u other than u. The concept of immediate dominator appears in many works such as [17] and [14]. We denote the immediate dominator w of vertex u by $id(u) = w$. For each vertex u in G $(u \neq s)$, there is a single vertex w which dominates immediately u. The graph $T_{id} = (V_{id}, A_{id})$ where $V_{id} = V$ and $A_{id} = \{id(u) \to u :$ for all u different from s in $V\}$ is a tree rooted in s, called the *dominator tree* of G. It is well known that a vertex w dominates a vertex u if and only if w is an ancestor of u in T_{id}, that is, the dominator tree of G is a compact representation of the dominator set for each vertex in G. In Fig. 2, we illustrate the dominator tree of the flowgraph in Fig. 1.

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