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# On the broadcast independence number of caterpillars

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ABSTRACT

Let $G$ be a simple undirected graph. A broadcast on $G$ is a function $f : V(G) \to \mathbb{N}$ such that $f(v) \leq e_G(v)$ holds for every vertex $v$ of $G$, where $e_G(v)$ denotes the eccentricity of $v$ in $G$, that is, the maximum distance from $v$ to any other vertex of $G$. The cost of $f$ is the value $\mathrm{cost}(f) = \sum_{v\in V(G)} f(v)$. A broadcast $f$ on $G$ is independent if for every two distinct vertices $u$ and $v$ in $G$, $d_G(u, v) > \max\{f(u), f(v)\}$, where $d_G(u, v)$ denotes the distance between $u$ and $v$ in $G$. The broadcast independence number of $G$ is then defined as the maximum cost of an independent broadcast on $G$.

In this paper, we study independent broadcasts of caterpillars and give an explicit formula for the broadcast independence number of caterpillars having no pair of adjacent trunks, a trunk being an internal spine vertex with degree 2.



## 1. Introduction

All the graphs we consider in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph $G$, respectively.

For any two vertices $u$ and $v$ of $G$, the *distance* $d_G(u, v)$ between $u$ and $v$ in $G$ is the length (number of edges) of a shortest path joining $u$ and $v$. The *eccentricity* $e_G(v)$ of a vertex $v$ in $G$ is the maximum distance from $v$ to any other vertex of $G$. The minimum eccentricity in $G$ is the *radius* $\mathrm{rad}(G)$ of $G$, while the maximum eccentricity in $G$ is the *diameter* $\mathrm{diam}(G)$ of $G$. Two vertices $u$ and $v$ with $d_G(u, v) = \mathrm{diam}(G)$ are said to be *antipodal*.

A function $f : V(G) \to \{0, \dots, \mathrm{diam}(G)\}$ is a *broadcast* if for every vertex $v$ of $G$, $f(v) \leq e_G(v)$. The value $f(v)$ is called the *$f$-value of $v$*. Given a broadcast $f$ on $G$, an *$f$-broadcast vertex* is a vertex $v$ with $f(v) > 0$. The set of all $f$-broadcast vertices is denoted $V_f^+$. If $u \in V_f^+$ is a broadcast vertex, $v \in V(G)$ and $d_G(u, v) \leq f(u)$, we say that $u$ *$f$-dominates* $v$. In particular, every $f$-broadcast vertex $f$-dominates itself. The *cost* $\mathrm{cost}(f)$ of a broadcast $f$ on $G$ is given by

$$\mathrm{cost}(f) = \sum_{v\in V(G)} f(v) = \sum_{v\in V_f^+} f(v).$$

A broadcast $f$ on $G$ is a *dominating broadcast* if every vertex of $G$ is $f$-dominated by some vertex of $V_f^+$. The minimum cost of a dominating broadcast on $G$ is the *broadcast domination number* of $G$, denoted $\gamma_b(G)$. A broadcast $f$ on $G$ is an *independent broadcast* if every $f$-broadcast vertex is $f$-dominated only by itself. The maximum cost of an independent broadcast on $G$ is the *broadcast independence number* of $G$, denoted $\beta_b(G)$. An independent broadcast on $G$ with cost $\beta$ is

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an independent $\beta$-*broadcast*. An independent $\beta_b(G)$-broadcast on $G$ is an *optimal* independent broadcast. Note here that any optimal independent broadcast is necessarily a dominating broadcast.

The notions of broadcast domination and broadcast independence were introduced by D.J. Erwin in his Ph.D. thesis [9] under the name of *cost domination* and *cost independence*, respectively. During the last decade, broadcast domination has been investigated by several authors, see e.g. [1–3,5–7,10,12–17], while independent broadcast domination has attracted much less attention.

In particular, Seager considered in [16] broadcast domination of caterpillars. She characterized caterpillars with broadcast domination number equal to their domination number, and caterpillars with broadcast domination number equal to their radius. Blair, Heggernes, Horton and Manne proposed in [1] an $O(nr)$-algorithm for computing the broadcast domination number of a tree of order $n$ with radius $r$.

However, determining the independent broadcast number of trees seems to be a difficult problem. We propose in this paper a first step in this direction, by studying a subclass of the class of caterpillars. Recall that a caterpillar is a tree such that deleting all its pendent vertices leaves a simple path called the spine. The subclass we will consider is the subclass of caterpillars having no pair of adjacent trunks, a trunk being an internal spine vertex with degree 2.

We now review a few results on independent broadcast numbers. Let $G$ be a graph and $A \subset V(G)$, $|A| \geq 2$, be a set of pairwise antipodal vertices in $G$. The function $f$ defined by $f(u) = \text{diam}(G) - 1$ for every vertex $u \in A$, and $f(v) = 0$ for every vertex $v \notin A$, is clearly an independent $|A|(\text{diam}(G) - 1)$-broadcast on $G$.

**Observation 1** (*Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [8]*)**.** *For every graph $G$ of order at least 2 and every set $A \subset V(G)$, $|A| \geq 2$, of pairwise antipodal vertices in $G$, $\beta_b(G) \geq |A|(\text{diam}(G) - 1)$. In particular, for every tree $T$, $\beta_b(T) \geq 2(\text{diam}(G) - 1)$.*

An independent broadcast $f$ on a graph $G$ is *maximal independent* if there is no independent broadcast $f' \neq f$ such that $f'(v) \geq f(v)$ for every vertex $v \in V(G)$. In [9], D.J. Erwin proved the following result (see also [8]).

**Theorem 2** (*Erwin [9]*)**.** *Let $f$ be an independent broadcast on $G$. If $V_f^+ = \{v\}$, then $f$ is maximal independent if and only if $f(v) = e_G(v)$. If $|V_f^+| \geq 2$, then $f$ is maximal independent if and only if the following two conditions are satisfied:*

1. *$f$ is dominating, and*
2. *for every $v \in V_f^+$, $f(v) = \min\left\{d_G(v, u) :\ u \in V_f^+ \setminus \{v\}\right\} - 1$.*

Erwin proved that $\beta_b(P_n) = 2(n - 2) = 2(\text{diam}(P_n) - 1)$ for every path $P_n$ of length $n \geq 3$ [9]. In [4], Bouchemakh and Zemir determined the independent broadcast number of square grids.

**Theorem 3** (*Bouchemakh and Zemir [4]*)**.** *Let $G_{m,n}$ denote the square grid with $m$ rows and $n$ columns, $m \geq 2$, $n \geq 2$. We then have:*

1. *$\beta_b(G_{m,n}) = 2(m + n - 3) = 2(\text{diam}(G_{m,n}) - 1)$ if $m \leq 4$,*
2. *$\beta_b(G_{5,5}) = 15$, $\beta_b(G_{5,6}) = 16$, and*
3. *$\beta_b(G_{m,n}) = \left\lceil \frac{mn}{2} \right\rceil$ for every $m, n$, $5 \leq m \leq n$, $(m, n) \neq (5, 5), (5, 6)$.*

In this paper, we determine the broadcast independence number of caterpillars having no pair of adjacent trunks. The paper is organized as follows. We introduce in the next section the main definitions and a few preliminary results on independent broadcasts of caterpillars. We then consider in Section 3 the case of caterpillars having no pair of adjacent trunks and prove our main result, which gives an explicit formula for the broadcast independence number of such caterpillars. We finally propose a few directions for future research in Section 4.

## 2. Preliminaries

Let $G$ be a graph and $H$ be a subgraph of $G$. Since $d_H(u, v) \geq d_G(u, v)$ for every two vertices $u, v \in V(H)$, every independent broadcast $f$ on $G$ satisfying $f(u) \leq e_H(u)$ for every vertex $u \in V(H)$ is an independent broadcast on $H$. Hence we have:

**Observation 4.** *If $H$ is a subgraph of $G$ and $f$ is an independent broadcast on $G$ satisfying $f(u) \leq e_H(u)$ for every vertex $u \in V(H)$, then the restriction $f_H$ of $f$ to $V(H)$ is an independent broadcast on $H$.*

A *caterpillar of length* $k \geq 0$ is a tree such that removing all leaves gives a path of length $k$, called the *spine*. Following the terminology of [16], a non-leaf vertex is called a *spine vertex* and, more precisely, a *stem* if it is adjacent to a leaf and a *trunk* otherwise. A leaf adjacent to a stem $v$ is a *pendent neighbour* of $v$. We will always draw caterpillars with the spine on a horizontal line, so that we can speak about the leftmost of rightmost spine vertex of a caterpillar.

Note that a caterpillar of length 0 is nothing but a star $K_{1,n}$, for some $n \geq 1$. The independent broadcast number of a star is easy to determine.

**Observation 5.** *For every integer $n \geq 1$, $\beta_b(K_{1,n}) = n$.*