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Recent results on containment graphs of paths in a tree

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ABSTRACT

In this paper, motivated by the questions posed by Spinrad in Spinrad (2003) and Golumbic and Trenk (2004), we investigate those posets that admit a containment model mapping vertices into paths of a tree and their comparability graphs, named *CPT* posets and *CPT* graphs, respectively. We present a necessary condition to be *CPT* and prove it is not sufficient. We provide further examples of *CPT* posets P whose dual P^d is non *CPT*. Thus, we introduce the notion of *dually-CPT* and *strong-CPT* posets. We demonstrate that, unlike what happens with posets admitting a containment model using interval of the line, the dimension and the interval dimension of *CPT* posets is unbounded. On the other hand, we find that the dimension of a *CPT* poset is at most the number of leaves of the tree used in the containment model. We give a characterization of *CPT* (also *dually-CPT* and *strong-CPT*) split posets by a family of forbidden subposets. We prove that every tree is *strong-CPT*.

1. Introduction and previous results

A *partially ordered set* or *poset* is a pair $\mathbf{P} = (X, P)$ where X is a finite non-empty set, whose elements are called *vertices*, and P is a reflexive, antisymmetric and transitive binary relation on X. As usual, we write $x \leq y$ in $\mathbf{P}$ for $(x, y) \in P$; and $x < y$ in $\mathbf{P}$ when $(x, y) \in P$ and $x \neq y$. If $x < y$ or $y < x$, we say that x and y are *comparable* in $\mathbf{P}$ and write $x \perp y$. The sets $\{x \in X : x < z\}$ and $\{x \in X : z < x\}$ are denoted by $D(z)$ and $U(z)$ respectively. We let $D[z] = D(z) \cup \{z\}$ and $U[z] = U(z) \cup \{z\}$. When $D(z) = \emptyset$, we say that z is a *minimal element* of $\mathbf{P}$; and z is *maximal* when $U(z) = \emptyset$. Two vertices z and z' are *twins* (*false twins*), if $D(z) \cup U(z) = D(z') \cup U(z')$ and z and z' are comparable (incomparable).

A *chain* in $\mathbf{P}$ is a subposet whose vertices are pairwise comparable. The *height* of $\mathbf{P}$ is one less than the number of vertices in its maximum chain.

The *restriction* of the relation P to a subset Y of X is denoted by $P(Y)$. We call $\mathbf{P}(Y)$ to the subposet $(Y, P(Y))$ of $\mathbf{P}$.

A *containment model* $M_{\mathbf{P}}$ of a poset $\mathbf{P} = (X, P)$ maps each element x of X into a set M_x in such a way that $x < y$ in $\mathbf{P}$ if and only if M_x is a proper subset of M_y, i.e.

$$x < y \text{ in } \mathbf{P} \Leftrightarrow M_x \subset M_y.$$

We identify the containment model $M_{\mathbf{P}}$ with the set family $(M_x)_{x \in X}$. Notice that a containment model can always be obtained by mapping each vertex x into the set $D[x]$. We say that a model is *injective* when no two vertices are mapped into a same set.

Many classes of posets, grouped together under the generic name of *geometric containment orders*, have been defined by imposing geometric conditions to the sets in which the elements of the poset are mapped: for example, they may be intervals

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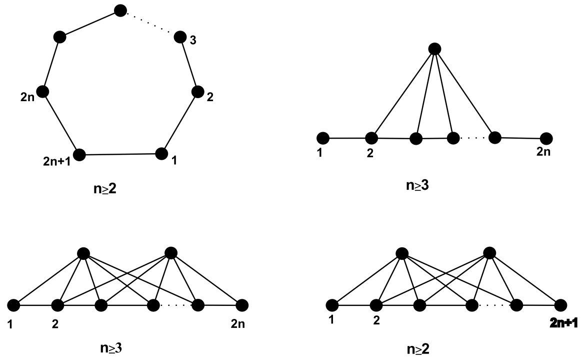

Fig. 1. These graphs together with the complements of the graphs in Fig. 2 constitute the family of minimal forbidden induced subgraphs for comparability graphs.

of the line, angular regions in the plane, d-boxes in the d-Euclidean space, d-spheres in the d-Euclidean space [4,9,16,20]. In [9], it is proved that each poset admits a containment model using subtrees of a star (a tree with a unique vertex with degree greater than one). As reported in [9], Corneil and Golumbic (see [8]) considered those posets that admit a containment model mapping vertices into paths of a tree, and their comparability graphs, called *CPT posets* and *CPT graphs*, respectively. They observed that the 8-wheel W_8 has one of its transitive orientations being a *CPT* poset (where the central vertex is a sink, i.e., its path is contained in each of the other 8 paths), but when reversing the orientations of the edges (the dual, where the central vertex is a source), it is not a *CPT* poset. Their same argument applies to every wheel W_{2k} for $k \geq 3$. Spinrad in [14] and Golumbic and Trenk in [10] called for investigating the properties of *CPT* posets and *CPT* graphs. In this paper, we have initiated such a study, presenting new results on the topic.

The comparability graph $G_{\mathbf{P}}$ of a poset $\mathbf{P} = (X, P)$ is the simple graph with vertex set $V(G_{\mathbf{P}}) = X$ and edge set $E(G_{\mathbf{P}}) = \{xy : x \perp y\}$. We say that two posets are *associated* if their comparability graphs are isomorphic. A graph G is a *comparability graph* if there exists some poset $\mathbf{P}$ such that $G = G_{\mathbf{P}}$.

A *transitive orientation* $\overrightarrow{E}$ of a graph $G = (V, E)$ is an assignment of one of the two possible directions, $\overrightarrow{xy}$ or $\overrightarrow{yx}$, to each edge $xy \in E$ in such a way that if $\overrightarrow{xy} \in \overrightarrow{E}$ and $\overrightarrow{yz} \in \overrightarrow{E}$ then $\overrightarrow{xz} \in \overrightarrow{E}$. The graphs whose edges can be transitively oriented are exactly the comparability graphs [6]. Furthermore, given a transitive orientation $\overrightarrow{E}$ of a graph $G = (V, E)$, we let $\mathbf{P}_{\overrightarrow{E}}$ denote the poset $(V, P_{\overrightarrow{E}})$ where $u < v$ in $\mathbf{P}_{\overrightarrow{E}}$ if and only if $\overrightarrow{uv} \in \overrightarrow{E}$. The comparability graph of $\mathbf{P}_{\overrightarrow{E}}$ is G. Thereby, the transitive orientations of G are put in one-to-one correspondence with the posets whose comparability graphs are G.

Gallai provides the following characterization of comparability graphs by a family of minimal forbidden induced subgraphs.

Theorem 1 ([5]). *A graph is a comparability graph if and only if none of its induced subgraphs is isomorphic to a graph in Fig. 1 or to the complement of a graph in Fig. 2.*

For further information on comparability graphs see [2,7,16].

Dushnik and Miller defined the *dimension of a poset* $\mathbf{P}$, denoted by $dim(\mathbf{P})$, as the minimum number of linear orders whose intersection is $\mathbf{P}$ [3]. Trotter et al. proved that if $\mathbf{P}$ and $\mathbf{P}'$ are associated posets then $dim(\mathbf{P}) = dim(\mathbf{P}')$, leading to the definition of *dimension of a comparability graph* [19,16].

The *dual* of a poset $\mathbf{P} = (X, P)$ is the poset $\mathbf{P}^d = (X, P^d)$ where $x < y$ in $\mathbf{P}^d$ if and only if $y < x$ in $\mathbf{P}$. Notice that $\mathbf{P}$ and $\mathbf{P}^d$ are associated and, obviously, $dim(\mathbf{P}) = dim(\mathbf{P}^d)$.

In [3], it was proved that $dim(\mathbf{P}) \leq 2$ if and only if $\mathbf{P}$ admits a containment model mapping vertices into intervals of the line. Therefore, posets with dimension at most 2 also appear in the literature as *containment orders of intervals*, we will write *CI* posets for short. Comparability graphs of interval containment orders, or *CI* graphs, have been widely studied and characterized in different ways.

Theorem 2 ([3,13]). *The following statements are equivalent.*

1. *G is a CI graph.*
2. *G is a comparability graph with $dim(G) \leq 2$.*
3. *G and its complement $\overline{G}$ are comparability graphs.*
4. *G is a permutation graph [2].*

The previous theorem together with Gallai's characterization of comparability graphs provides a characterization of *CI* graphs by induced forbidden subgraphs. In addition, observe the following simple result:

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