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On the contour of bipartite graphs ☆

D. Artigas [a,*], R. Sritharan [b]

[a] Instituto de Ciência e Tecnologia, Universidade Federal Fluminense, Brazil
[b] Computer Science Department, The University of Dayton, Dayton OH 45469, USA

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ABSTRACT

A vertex of a connected graph is a contour vertex provided the eccentricity of the vertex is at least as large as that of each of its neighbors. We consider the question of whether the set S of contour vertices of a connected graph is geodetic, i.e., whether every vertex of the graph lies on a shortest path (geodesic) between some pair of vertices in S. In general, it is known that when long induced cycles are forbidden (for chordal graphs) the answer is in the affirmative, but otherwise (even for weakly chordal graphs) the answer is in the negative. For bipartite graphs, it is known that when long induced cycles are allowed, the answer is in the negative. In contrast, we show that when long induced cycles are forbidden in bipartite graphs, namely for chordal bipartite graphs, the answer is in the affirmative. Our result also shows that while the answer is in the negative for bipartite graphs and weakly chordal graphs, for a graph that is both bipartite and weakly chordal, the answer is in the affirmative.

1. Introduction

In this work all graphs are finite, simple, and connected, unless noted otherwise. Let $G = (V, E)$ be a graph. A family $\mathcal{C}$ of subsets of V is a *graph convexity*, or convexity, if it satisfies two properties: (i) V, $\emptyset$ belong to $\mathcal{C}$; (ii) $\mathcal{C}$ is closed under intersections. The elements of $\mathcal{C}$ are called *convex sets*. In the recent past, several papers have been published on graph convexity [1,4,10,14–16,18,19,22,23]. For general information about convexity see [26].

In this work we consider the concept of *geodesic convexity*, where a set of vertices S is *convex* if it is formed by all vertices lying on a geodesic between any pair of vertices of S. A set of vertices S is *geodetic* if every vertex of the graph belongs to a geodesic between some pair of vertices in S. Geodesic convexity has been studied in different contexts such as convex partitions, geodetic sets, geodetic number, hull number, and convexity number [2,6,12,13]. For a survey about geodesic convexity see [25].

In [8], the authors investigated whether certain specific sets of vertices in a graph were geodetic; here, the authors were concerned with what they considered to be "boundary vertices" in graphs. In particular, the contour $Ct(G)$ of a graph G was defined in [8] as follows: $v \in V(G)$ belongs to $Ct(G)$ provided eccentricity of v is at least as large as that of each of the neighbors of v. The issue of whether the set of contour vertices of a graph always constitutes a geodetic set was then studied. Then, and subsequently, the problem has been studied for several structured classes of graphs [3,7,6,17]. We frequently use *contour of a graph* to refer to the set of contour vertices of the graph.

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* Corresponding author.
E-mail addresses: daniloartigas@puro.uff.br (D. Artigas), rsritharan1@udayton.edu (R. Sritharan).

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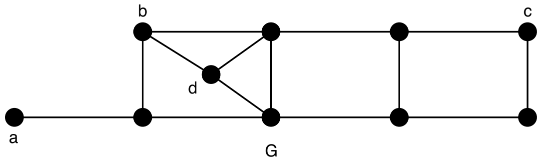

Fig. 1. A perfect (weakly chordal) graph whose contour is not geodetic [7].

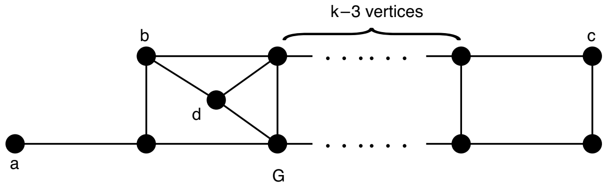

Fig. 2. Graph, with diameter $k \geq 5$, whose contour is not geodetic [3]. The corresponding pairs of vertices from the $k-3$ vertices on "top" and the $k-3$ at the "bottom" are connected by a matching.

Given graphs G and H, by G *contains* H we mean G contains H as an induced subgraph. Let P_k, $k \geq 1$, denote an induced path on k vertices. C_k, $k \geq 4$, denotes an induced cycle on k vertices. A $2K_2$ is the complement of a C_4. A C_k, $k \geq 5$, is a *hole*, and an *antihole* is the complement of a hole. A *house* is the complement of a P_5. A *domino* is the bipartite graph obtained from a cycle on six vertices by adding exactly one chord. A *gem* is the complement of the disjoint union of a P_1 and a P_4. A graph is *chordal* if it does not contain any C_k, $k \geq 4$. Finally, a graph is *weakly chordal* if it does not contain any holes or antiholes.

It was proved in [8] that the contour of *distance hereditary graphs*, namely graphs without a house, hole, domino, or a gem, is a geodetic set. An example was also given in [8] to show that the contour of a graph, in general, need not be geodetic; specifically, this example contains a hole on 7 vertices and hence is not perfect. Subsequently, the example in Fig. 1 was given in [7] to show that there exists a permutation graph, and hence a perfect graph, whose contour is not geodetic. On the positive side, it was shown in [7] that the contour of a chordal graph is geodetic. Recently, the results on distance hereditary graphs and chordal graphs were generalized in [17] to show that the contour of an *hhd-free graph*, namely a graph without a house, hole, or a domino, is geodetic. The class of weakly chordal graphs is even more general than the class of hhd-free graphs. However, the example in Fig. 1 shows that the contour of a weakly chordal graph need not be geodetic.

The question of whether the contour of a graph belonging to a class of graphs is geodetic was posed in [7] for the classes of cochordal (complements of chordal), bipartite, and parity graphs. In [3], the question was answered in the affirmative for cochordal graphs and in the negative for each of bipartite and parity graphs. More specifically, the example from [7] was generalized in [3] into a family of graphs in each of which the contour is not geodetic. The thrust of this family, shown in Fig. 2, is to show that there exists a family of general graphs with diameter $k \geq 5$ such that the contour of any member of the family is not geodetic. In contrast, it was shown in [3] that for every graph with diameter at most 4, the contour is geodetic. It was further shown in [3] that in the case of bipartite graphs, if the diameter is at most 7, then the contour is geodetic, but in contrast, there exists a family of bipartite graphs with diameter $8 + k$, $k \geq 0$, for each member of which the contour is not geodetic; we refer the reader to Fig. 3.

As far as whether the contour of a graph is geodetic is concerned, every negative example known thus far contains a house or a hole. In the case of bipartite graphs, even the smallest of the members of the families given in [3] contains induced cycles on 8 or more vertices; thus, when long induced cycles are allowed in bipartite graphs, the answer is in the negative [3]. In summary, it is known in general that when long induced cycles are forbidden (for chordal graphs) the answer is in the affirmative, but otherwise (even for weakly chordal graphs) the answer is in the negative. For bipartite graphs, it is known that when long induced cycles are allowed, the answer is in the negative. In contrast, we show that when long induced cycles are forbidden in bipartite graphs, namely for *chordal bipartite graphs*, the answer is in the affirmative; a bipartite graph is chordal bipartite if it does not contain any C_{2k}, $k \geq 3$. Our result also shows that while the answer is in the negative for each of bipartite graphs and weakly chordal graphs, for a graph that is both bipartite and weakly chordal (for a chordal bipartite graph), the answer is in the affirmative.

We note that it has recently been proved [24] that the contour of a bridged graph is geodetic. Our techniques are different from those employed in a majority of the previous results [7,8,17,24] and we rely on structural properties of chordal bipartite graphs to make inductive arguments; we elaborate on this further in Section 3. In order to discuss this, we first present in Section 3 an inductive proof to show that the contour of a chordal graph is geodetic using its structural properties.

2. Preliminaries

Let $G = (V, E)$ be a graph with vertex set V and edge set E, where $|V| = n$ and $|E| = m$. For $S \subseteq V$, $G[S]$ denotes the subgraph of G *induced* by S.

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