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Axiomatic characterization of the center function. The case of non-universal axioms

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ABSTRACT

The center function is defined on a connected graph G, where the input is any finite sequence of vertices of G and the output is the set of all vertices that minimize the maximum distance to the entries of the input. If the input is a sequence containing each vertex of G once, then the output is just the classical center of G. In the axiomatic approach, one wants to establish a set of properties or consensus axioms that characterize the function. We refer to an axiom as a *universal axiom* if the center function satisfies this axiom on any connected graph. In a previous paper, our focus was on classes of graphs on which we were able to characterize the center function in terms of universal axioms. In this paper, the focus is on classes of graphs on which these universal axioms do not characterize the center function. We introduce *non-universal axioms* that, together with some universal axioms, provide new characterizations of the center function: on cocktail party graphs, on complete bipartite graphs, and on block graphs.

1. Introduction

In 2001 McMorris, Roberts and Wang [25] initiated the axiomatic study of the center function on graphs. Let G be a finite connected graph. The center function has as input a profile, i.e. a finite sequence of vertices π of G. The output is the set of vertices that minimize the maximum distance to the entries of π. Thus the center function is an instance of the classical and well studied minimax problem in location theory, see e.g. [11]. With respect to the axiomatic study, the center function has not yet attracted much attention, in contrast with another location function known as the median function. This function also has π as input, but now the output is the set of vertices that minimize the average distance to the entries of π, or, equivalently, minimize the sum of the distances to the entries of π (the classical minisum problem).

In [10] we pointed at the similarities and the differences between the center function and the median function. A common aspect is the following. The median function and the center function are instances of the ℓ_p-function studied in [20]. This function uses the ℓ_p-norm to measure the distance between a vertex and a profile. As is well known, the ℓ_1-norm measures the sum of the distances to a vertex, and minimizing this is precisely what the median function does. The ℓ_∞-norm measures the maximum distance to a vertex, and minimizing this is precisely what the center function does. We refer the reader to [10] for more information on this. So both the median function and the center function are ℓ_p-functions. On the other hand,

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they represent the extreme opposites within the spectrum of the ℓ_p-functions, which also explains the striking differences between the two.

The median function has been studied extensively, see e.g. [8,9,18,19,21–24,26,30,36]. Most of these papers consider the median function as a consensus function. Such a function models how a set of agents wants to achieve consensus in a rational way. The input is certain information on the agents (like their location in the graph), and the output is the set of locations upon which the agents agree. The rationality of the process is guaranteed by certain properties, usually called consensus axioms. These consensus axioms should be as simple and as appealing as possible. Thus we get the problem of characterizing a consensus function by a set of consensus axioms. The axiomatic study of social choice can be traced back to 1950 and the seminal work of Kenneth Arrow [1,2]. Already in 1952, May [17] gave an axiomatic characterization of the simple majority rule. Another seminal publication is the work of Kemeny and Snell [16], where, maybe for the first time, the axiomatic approach was combined with consensus terminology. Holzman [15] was the first to study location functions from the perspective of consensus. For more information see also [3,4,7,12,32].

McMorris, Roberts and Wang, cited above, obtained an axiomatic characterization of the center function on trees. They proved that it is the unique consensus function on a tree that satisfies four very reasonable axioms: a version of anonymity, middleness, a version of consistency, and redundancy. We explain these axioms below in more detail. Center sets in various classes of graphs were collected in [5], see that paper also for other references on the centers of graphs. Recently, using an interesting, slightly different approach, Seif [33] studied the center function on 3-sun-free chordal graphs.

The median function satisfies some simple and appealing axioms on all connected graphs: anonymity, betweenness and consistency. In [30] it was proved that, on the class of median graphs, the median function is the unique function satisfying these three simple axioms. This result was the motivation in [19] to ask the following questions:

(*i*) On which classes of graphs is the median function the unique function satisfying the three basic axioms of anonymity, betweenness and consistency?

(*ii*) If these axioms do not yet characterize the median function on some class of graphs $\mathcal{G}$, what extra axioms are needed to characterize the median function on $\mathcal{G}$?

Of course, similar questions can be asked for the center function, where we consider the relevant axioms in this case. The analogue of the first question for the center function is considered in [10]. In that paper consensus axioms were collected that are satisfied by the center function on any connected graph. Such axioms are called 'universal axioms' for the center function. Several classes of graphs were presented, on which the center function is actually the unique function that satisfies these universal axioms.

In the present paper we focus on the analogue of the second question. Here we will give classes of graphs, on which there are more than one consensus function satisfying these universal axioms. Then we will search for non-universal axioms that hold for the center function on specific classes, but not on all connected graphs. Finally, we characterize the center function using a combination of universal axioms and non-universal axioms.

In Section 2 we set the stage. In Section 3.1 we list the universal axioms. In Section 3.2 we collect some class-specific axioms. In Section 4 we study graph classes on which the center function needs non-universal axioms, viz. the cocktail party graphs, the complete bipartite graphs, and finally the block graphs. We may summarize the main results of our paper as follows. For the definitions of all notation and axioms, we refer to Sections 2 and 3.

Theorem. *Let G be a connected graph with vertex set* $V = \{v_1, v_2, \ldots, v_n\}$. *Let L be a consensus function on G such that L satisfies* (*PI*), (*M*)*, and* (*Pre-C*)*.*

(*i*) *If G contains a dominating vertex, then* $L = Cen$*.*

(*ii*) *If G is a cocktail party graph and L satisfies* (*Inc*) *for the particular profile* $\pi = (v_1, v_2, \ldots, v_n)$*, then* $L = Cen$*.*

(*iii*) *If G is the complete bipartite graph* $K_{m,n}$ *with* $m, n \geq 2$ *and L satisfies* (*Inc*) *for all profiles* π *such that* $|\pi| \geq 3$*, then* $L = Cen$*.*

(*iv*) *If G is a block graph and L satisfies* (*R*)*, then* $L = Cen$*.*

Here (*PI*), (*M*), and (*Pre-C*) are universal axioms that hold for *Cen* on any connected graph, see Section 3.1. The axioms (*Inc*) and (*R*) are non-universal axioms that hold only on specific classes, see Section 3.2. Result (*i*) is from [10]. We list it here to highlight the difference between the results in that paper and in the present paper. Results (*ii*) and (*iii*) are Theorems 5 and 16, respectively. Result (*iv*) is Theorem 20, which extends the result on trees in the seminal paper of McMorris, Roberts, and Wang [25]. Note that the three universal axioms (*PI*), (*M*), and (*Pre-C*) do not suffice to characterize *Cen* on the graphs in (*ii*), (*iii*), and (*iv*). In all these cases we provide examples of consensus functions that satisfy the three universal axioms, but are not *Cen*.

2. Preliminaries

Throughout this paper G is a finite, connected, simple, loopless graph with vertex set V. We denote the set of neighbors of a vertex u by $N(u)$, and write $N[u] = N(u) \cup \{u\}$ for the *closed neighborhood* of u. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest u, v-path or u, v-geodesic. The *interval* $I_G(u, v)$ between u and v in G is the set of vertices on the geodesics between u and v, that is,

$$I_G(u, v) = \{w \mid d(u, w) + d(w, v) = d(u, v)\}.$$