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Forcing and anti-forcing edges in bipartite graphs

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ABSTRACT

Let G be a connected bipartite graph with a perfect matching and the minimum degree at least two. The concept of an anti-forcing edge in G was introduced by Li in Li (1997). One known generalized version for an anti-forcing edge is an anti-forcing set S, which is a set of edges of G such that the spanning subgraph $G - S$ has a unique perfect matching. In this paper, we introduce a new generalization of an anti-forcing edge: an anti-forcing path and an anti-forcing cycle. We show that the existence of an anti-forcing edge in G is equivalent to the existence of an anti-forcing path or an anti-forcing cycle in G. Then we show that G has an edge that is both forcing and anti-forcing if and only if G is an even cycle. In addition, e-anti-forcing paths and e-anti-forcing cycles in hexagonal systems are identified. The parallel concepts of forcing-paths and forcing-cycles of G are also presented.

1. Introduction

The notion of the forcing edge first appeared in a 1991 paper [7] on polyhexes by Harary, Klein and Živković, where they studied one fundamental aspect of a perfect matching M: a subset $S \subseteq M$ with the minimum cardinality that completely determines M, and the cardinality of such a subset S is called the forcing number of M. The same concept was introduced by Klein and Randić [8] in 1987 under the name innate degree of the freedom of a Kekulé structure. An edge of a connected graph is called a forcing edge (or, a forcing double edge) if it is contained in exactly one perfect matching of the graph. Forcing edges have been investigated intensively for hexagonal systems which are closely related to the study of molecule resonance structures in benzenoid hydrocarbons, where a hexagonal system (or, a polyhex) is a 2-connected plane bipartite graph such that each finite face is a unit hexagon. For a survey on the topic of forcing perfect matchings, the reader is referred to [1]. For related research on non-bipartite graphs, the readers may refer to the papers [14,18] and the references therein.

In 1997, Li [10] introduced the concept of an anti-forcing edge in the name of a forcing single edge when studying hexagonal systems. In the same paper, all hexagonal systems with a forcing single edge were characterized. An edge e of a graph G is called an anti-forcing edge (or, a forcing single edge) if the spanning subgraph $G - e$ has a unique perfect matching. A generalized version for an anti-forcing edge is an anti-forcing set S, which is a set of edges of G such that the spanning subgraph $G - S$ has a unique perfect matching. The anti-forcing number of a graph was introduced by Vukičević and Trinajstić [12] as the minimum cardinality of an anti-forcing set of the graph. A lot of research work has been done on anti-forcing numbers, for example, see [4–6,9,13,15,19].

In the current paper, graphs considered are connected bipartite graphs with a perfect matching and the minimum degree at least two. To further explore the concept of an anti-forcing edge e in such a graph G, we introduce the concepts of e-anti-forcing paths and e-anti-forcing cycles. We show that the existence of an anti-forcing edge in G is equivalent to the

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existence of an anti-forcing path or an anti-forcing cycle in G. This enables us to get the result that G has an edge that is both forcing and anti-forcing if and only if G is an even cycle. In addition, this allows us to identify e-anti-forcing paths and e-anti-forcing cycles in hexagonal systems. The parallel concepts of forcing-paths and forcing-cycles in this type of graphs are also presented.

2. Preliminaries

Graphs in our paper are simple and finite. In a graph G, the set of all vertices adjacent to a vertex v is called the neighborhood of v and denoted by $N_G(v)$. The vertex degree of v in G is the cardinality of $N_G(v)$ and denoted by $\deg_G(v)$. The minimum degree of G is the minimum vertex degree of all vertices in G and denoted by $\delta(G)$. A vertex of G is called a pendant vertex if it is a degree-1 vertex of G. Let α and β be positive integers. An edge of G is called an (α, β)-edge if its two end vertices have degrees α and β in G, respectively. A pendant edge of G is an $(\alpha, 1)$-edge of G. A matching edge deletion of an edge e of G [7] is to delete the end vertices of e together with their incident edges including the edge e itself. A perfect matching (or, 1-factor) of a graph is a set of disjoint edges that covers all vertices of the graph. The key idea for an anti-forcing edge or a forcing edge of a graph G is based on the uniqueness of a perfect matching for a certain subgraph: deleting an anti-forcing edge of G results in a spanning subgraph that has a unique perfect matching, and a matching edge deletion of a forcing edge of G results in either an empty graph or an induced subgraph with a unique perfect matching. Let $e = uv$ be an edge of G with two end vertices u and v. The spanning subgraph of G obtained by removing an edge e is denoted by $G - e$, and the induced subgraph of G obtained by a matching edge deletion of e is denoted by $G - \{u, v\}$. In general, for a subset U of $V(G)$, we use $G - U$ to denote the induced subgraph of G obtained by removing all vertices from U together with their incident edges. A bipartite graph is a simple and finite graph whose vertices can be colored properly with two colors such that any two adjacent vertices have different colors. In the rest of this paper, all graphs considered are bipartite graphs with a perfect matching, unless specified otherwise. We also assume that vertices of a bipartite graph are properly colored in black and white. By Lemma 4.3.2 in [11], we have the following proposition.

Proposition 2.1 (*[11]*). *Any bipartite graph (can be disconnected) with a unique perfect matching has two degree-*1 *vertices that are in different colors.*

It follows that any bipartite graph with $\delta(G) > 1$ and a perfect matching has at least two perfect matchings. By the remark after Theorem 1 in [10], we have the following theorem.

Theorem 2.2 (*[10]*). *Let G be a connected bipartite graph with $\delta(G) > 1$. Then an edge $e = uv$ of G is an anti-forcing edge if and only if e is a $(2, 2)$-edge and G has two forcing edges different from e that are incident to u and v, respectively.*

By Theorem 2.2, we can see that for a connected bipartite graph G with the minimum degree at least two, the existence of an anti-forcing edge implies the existence of a forcing edge in G. But it is not true conversely. For example, the hexagonal system given in Fig. 1 has two forcing edges (two peripheral edges belonging to the hexagon centered at O) but no anti-forcing edges. The following result follows immediately from Theorem 2.2. It also appeared in a recent paper [6].

Corollary 2.3. *Every edge of a connected bipartite graph G is anti-forcing if and only if G is an even cycle.*

We provided the corresponding result of the above corollary for forcing edges [2], which generalizes a main result of Harary et al. in [7] from polyhexes to connected bipartite graphs.

Theorem 2.4 (*[2]*). *Every edge of a connected bipartite graph G is forcing if and only if G is an even cycle or an edge.*

Proposition 2.5. *Assume that G is a simple and finite graph. Let G' be a subgraph of G obtained by a matching edge deletion of a pendant edge of G. Then G has a unique perfect matching if and only if G' is empty or G' has a unique perfect matching.*

We omit the proof of the above proposition since it is trivial. Using Propositions 2.1 and 2.5, we can obtain an algorithm to determine whether a bipartite graph with a pendant edge has a unique perfect matching or not by repeatedly performing matching edge deletions of pendant edges, which is similar to the algorithm given by Harary et al. in [7].

All hexagonal systems in this paper are assumed to be drawn in a position such that edges in one direction are vertical. Let H be a hexagonal system. Then a vertex of H belongs to at most three hexagons. A vertex of H is called an interior vertex if it is shared by three hexagons, and an exterior vertex otherwise. The periphery of H is the boundary cycle consisting of all its exterior vertices. Any edge on the periphery of H is called a peripheral edge of H, and any hexagon of H with a peripheral edge is called a peripheral hexagon of H. Two hexagons of H are said to be adjacent if they have a common edge. The inner dual H^* of a hexagonal system H is the graph each vertex of which corresponds to the center of a hexagon of H and two vertices are adjacent in H^* if the corresponding two hexagons are adjacent in H. A hexagonal system is called a hexagonal chain if its inner dual is a path or a single vertex. Furthermore, a hexagonal chain is said to be a linear hexagonal chain if its inner dual is a straight path or a single vertex, and a nonlinear hexagonal chain otherwise.

The following concepts (illustrated in Fig. 1) were introduced in [16] to characterize hexagonal systems whose resonance graph has a pendant edge, and later used in [17] to characterize hexagonal systems with a forcing edge.