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Recognition and characterization of unit interval graphs with integer endpoints

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ABSTRACT

We study those unit interval graphs having a model with intervals of integer endpoints and prescribed length. We present a structural result for this graph subclass which leads to a quadratic-time recognition algorithm, giving as positive certificate a model of minimum total length and as negative certificate a forbidden induced subgraph. We also present a quadratic-time algorithm to build, given a unit interval graph, a unit interval model with integer endpoints for which the interval length is as minimum as possible.

1. Introduction

A graph is an *interval graph* if there exists a bijection between a family of open intervals in the real line, and its vertex set, such that two vertices are adjacent if and only if their corresponding intervals intersect. Such a family of intervals is called an *interval model* of the graph. Let $G = (V, E)$ be a graph, $H = (V', E')$ is said to be an *induced subgraph* of G if $V' \subseteq V$ and $E' = \{uv \in E : u, v \in V'\}$. Given a collection of graphs $\mathcal{H}$, G is defined to be $\mathcal{H}$*-free* if for any graph $H \in \mathcal{H}$, G does not contain an induced H. If $\mathcal{H}$ is a set with a single element H, we just use H-free for short. Interval graphs were characterized by forbidden induced subgraphs in a celebrated paper of Boland and Lekkerkerker [6]. Graphs in this class can be recognized in linear-time (see e.g. [5]). A *unit interval graph* is an interval graph having a model with all its intervals of the same length. Such an interval model is called a *unit interval model*. Roberts proves that unit interval graphs are exactly those claw-free interval graphs [9]. Unit interval graphs can be recognized in linear-time. For instance, a direct and simple linear-time recognition algorithm for unit interval graphs based on BFS search can be found in [1]; in case the input graph is a unit interval, the algorithm outputs a unit interval model in which every endpoint is an integer ranging from 0 to n^2.

For definitions and concepts not defined here, see [11]. Two adjacent vertices are *true twins*, or simply *twins*, if they have the same closed neighborhood, where two vertices are *neighbors* if they are adjacent. Given an edge vx and a vertex y of G, if x, y are twin vertices in $G - \{v\}$ but not in G, that is, $N_G[x] \Delta N_G[y] = \{v\}$, we will say that v *distinguishes* x from y. Roberts [9]

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shows that the ordering of the left endpoints of any unit interval model of a connected unit interval graph without twins is unique up to reversal, called a *canonical ordering*. Throughout this paper, the results given will concern finite, undirected and connected graphs. Every connected component can be analyzed separately for disconnected graphs; thus, we will only consider the connected graphs. Extending the results for unconnected graphs is trivial. Given an interval I, denote by $\ell(I)$ and $\mathrm{r}(I)$ the left endpoint and the right endpoint of I, respectively. For a twin-free connected graph, let $\mathcal{M} = \{I_1, \ldots, I_n\}$ be a unit interval model indexed in the canonical order, where I_1 has the leftmost left endpoint and I_n has the rightmost one, we define the *total length* of $\mathcal{M}$ as $|\mathcal{M}| = \mathrm{r}(I_n) - \ell(I_1)$. We also define $\mathrm{Left}(\mathcal{M}) = \ell(I_1)$ and $\mathrm{Right}(\mathcal{M}) = \mathrm{r}(I_n)$. For a positive integer k, an open (resp. closed) interval with integer endpoints and length k is a (k)-*interval* (resp. $[k]$-*interval*). Given a graph G and a positive integer k, we say that G is a (k)-*interval graph* (resp. a $[k]$-*interval graph*) if G admits a model $\mathcal{M}$ of (k)-intervals (resp. $[k]$-intervals). This unit interval model is called (k)-*interval model* (resp. $[k]$-*interval model*). Notice that every unit interval graph has a (k)-interval model for some k. Therefore, given a unit interval model $\mathcal{M}$, it is interesting to find another unit interval model $\mathcal{M}'$ such that k and $\left|\mathcal{M}'\right|$ are minimum. This problem was studied in different contexts. Continuing with some ideas developed by Pirlot in [8], Mitas presents a linear-time algorithm that constructs a (k)-interval model with a minimum k for a given unit interval model [7]. This algorithm was developed in the context of the study of interval semiorders representable with intervals of a given length k. Moreover, those semiorders representable with intervals of length k were characterized by forbidden suborders for every integer k. Recently, Soulignac [10] reported a flaw in Mitas' algorithm [7], showed examples in which the integer k, found by the algorithm, is not minimum and the flaw is fixed but the complexity becomes quadratic-time. The problem of finding a unit interval model with minimum k and minimum total length is also studied in the context of gene clusters as intersection of powers of paths [2].

In this work, we give a new approach to solve the following problems:

1. recognize if a graph is a (k)-interval graph for a given k and exhibit a (k)-interval model of minimum total length,
2. find the minimum k in which a graph is a (k)-interval graph.

More precisely, given a graph G and a nonnegative integer k, we develop an algorithm that determines (in quadratic-time) if G is a (k)-interval graph and if so, shows a (k)-interval model of minimum total length. Otherwise, it shows a forbidden induced subgraph contained in G (a negative certificate). In addition, we derive from this algorithm another one that, given a unit interval graph G and a unit interval model of it, finds the minimum k in which G is a (k)-interval graph. This algorithm also constructs a (k)-interval model. Furthermore, we present a characterization by forbidden induced subgraphs for the class of (k)-interval graphs for every integer k. Notice that recognizing a unit interval graph and obtaining a unit interval model is linear [1]; therefore, we shall assume that a unit interval model of an input graph is given whenever the input graph is a unit interval graph.

This paper is organized as follows. In Section 2, we present a characterization for the class of (k)-interval graphs for every integer k by forbidden induced subgraphs and a quadratic-time algorithm that decides for a given integer k whether a unit interval graph has a (k)-interval model or not. Finally, in Section 3, we present an algorithm that finds in quadratic-time a (k)-interval model of minimum k and minimum total length.

An extended abstract of this work was published in [3].

2. Structural characterization

In this section, we present a family of forbidden induced subgraphs for a (k)-interval graph. Our main result is a characterization of this class and an algorithm to exhibit a model of minimum total length.

Before presenting the main result of this section, we need to introduce new definitions. Given an interval $I = (a, b)$, we say that I is *left shifted* if it is replaced by the interval $(a-1, b-1)$, that is, if I is shifted one unit to the left. Given a (k)-interval model $\mathcal{M}$ of a (k)-interval graph G and $I \in \mathcal{M}$, we define $\mathcal{L}_k(I)$ as the minimum submultiset $\mathcal{S}$ of $\mathcal{M}$ such that left shifting all intervals in $S \cup I$ yields a new (k)-interval model of G and $\mathcal{L}_k(A)$ for some multiset A of (k)-intervals as $\cup_{I \in A} \mathcal{L}_k(I)$, where $\mathcal{L}_k(\emptyset) = \emptyset$. Let $\mathrm{RO}(I)$ be the multiset of the intervals of a (k)-interval model which overlap I one unit on the right endpoint and the emptyset otherwise, and $\mathrm{L}(I)$ be the multiset of the intervals of a (k)-interval model whose right endpoint coincides with the left endpoint of I and the emptyset otherwise. Fig. 1 shows $\mathrm{RO}(I)$ and $\mathrm{L}(I)$ of a certain interval $I = (k, 2k)$. If I is left shifted, $\mathrm{RO}(I)$ and $\mathrm{L}(I)$ must also be left shifted so that $\mathcal{M}$ is still a (k)-interval model of G. But, as $\mathrm{RO}(I)$ and $\mathrm{L}(I)$ will be left shifted, $\mathcal{L}_k(L(I))$ and $\mathcal{L}_k(RO(I))$ must also be left shifted; therefore, a recursive way of defining $\mathcal{L}_k : \mathcal{M} \to \mathcal{P}(\mathcal{M})$ is as follows:

$$\mathcal{L}_k(I) = \mathcal{L}_k(\mathrm{RO}(I)) \cup \mathcal{L}_k(\mathrm{L}(I)) \cup \mathrm{RO}(I) \cup \mathrm{L}(I),$$

where $\mathcal{P}(\mathcal{M})$ is the family of all submultisets $S \subseteq \mathcal{M}$. When the length k is clear in the context, we shall omit the subindex and use $\mathcal{L} = \mathcal{L}_k$.

Consider the (k)-interval model $\widetilde{\mathcal{M}}$ obtained from $\mathcal{M}$ left shifting all intervals in $\mathcal{L}(I) \cup \{I\}$ for some $I \in \mathcal{M}$. Notice that $\widetilde{\mathcal{M}}$ is a (k)-interval model of the same graph as $\mathcal{M}$. In fact, $\mathcal{L}(I)$ is the minimum multiset of intervals that has to be left shifted when translating I one unit to the left so as to preserve adjacencies. Suppose $\mathcal{M}$ is a (k)-interval model with two coinciding intervals $I = I'$. The interval I belongs to $\mathcal{L}(I')$ if and only if there exists a sequence of intervals $I_1, I_2, \ldots, I_N$ in $\mathcal{L}(I') \cup \{I'\}$ such that $I_1 = I'$ and I_{i+1} equals either $L(I_i)$ or $RO(I_i)$ for each $1 \le i \le N-1$ and either $L(I_N) = I$ or $RO(I_N) = I$. Suppose that $L(I_N) = I$, the other case is analogous as noted in Remark 1. Given two integers a and b, $[\![a, b]\!]$ stands for the set formed by