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On the representation number of a crown graph

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ABSTRACT

A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. It is known (Kitaev and Pyatkin, 2008) that any word-representable graph G is k-word-representable for some k, that is, there exists a word w representing G such that each letter occurs exactly k times in w. The minimum such k is called G's representation number.

A crown graph (also known as a cocktail party graph) $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. In this paper, we show that for $n \geq 5$, $H_{n,n}$'s representation number is $\lceil n/2 \rceil$. This result not only provides a complete solution to the open Problem 7.4.2 in Kitaev and Lozin (2015), but also gives a negative answer to the question raised in Problem 7.2.7 in Kitaev and Lozin (2015) on 3-word-representability of bipartite graphs. As a byproduct, we obtain a new example of a graph class with a high representation number.

1. Introduction

Suppose that w is a word over some alphabet and x and y are two distinct letters in w. We say that x and y *alternate* in w if after deleting in w all letters but the copies of x and y we either obtain a word $xyxy\cdots$ (of even or odd length) or a word $yxyx\cdots$ (of even or odd length).

A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if xy is an edge in E. For example, the cycle graph on 4 vertices labelled by 1, 2, 3 and 4 in clockwise direction can be represented by the word 14213243.

There is a long line of research on word-representable graphs, which is summarized in the recently published book [8]. The roots of the theory of word-representable graphs are in the study of the celebrated Perkins semigroup [10,13] which has played a central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. However, the most interesting aspect of word-representable graphs from an algebraic point of view seems to be the notion of a semi-transitive orientation [6], which generalizes partial orders. It was shown in [6] that a graph is word-representable if and only if it admits a semi-transitive orientation.

More motivation points to study word representable graphs include the fact exposed in [8] that these graphs generalize several important classes of graphs such as *circle graphs* [4], 3-*colourable graphs* [1] and *comparability graphs* [12]. Relevance of word-representable graphs to scheduling problems was explained in [6] and it was based on [5]. Furthermore, the study of word-representable graphs is interesting from an algorithmic point of view as was explained in [8]. For example, the *Maximum Clique problem* is polynomial time solvable on word-representable graphs [8] while this problem is generally NP-complete [3]. Finally, word-representable graphs is an important class among other graph classes considered in the

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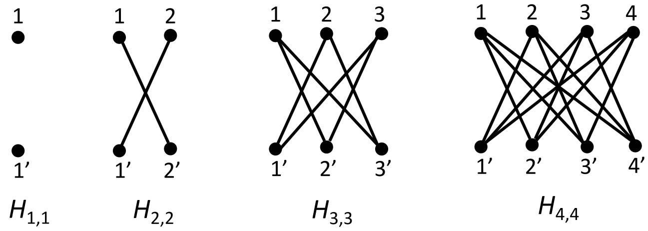

Fig. 1. The crown graph $H_{n,n}$ for $n = 1, 2, 3, 4$.

Table 1
Representing $H_{n,n}$ as a concatenation of n permutations.

n	Representation of $H_{n,n}$ by concatenation of n permutations
1	$11'1'1$
2	$12'21'21'12'$
3	$123'32'1'132'23'1'231'13'2'$
4	$1234'43'2'1'1243'34'2'1'1342'24'3'1'2341'14'3'2'$

literature that are defined using words. Examples of other such classes of graphs are *polygon–circle graphs* [11] and *word-digraphs* [2].

It was shown in [9] that if a graph G is word-representable then it is *k-word-representable* for some k, that is, G can be represented by a *k-uniform* word w, i.e. a word containing k copies of each letter. In such a context we say that w *k-represents* G. For example, the cycle graph on 4 vertices mentioned above can be 2-represented by the word 14213243. Thus, when discussing word-representability, one need only consider k-uniform words. A nice property of such words is that any cyclic shift of a k-uniform word represents the same graph [9]. (If a word $w = uv$ for two non-empty words u and v, then the word vu is a cyclic shift of w.) The minimum k for which a word-representable graph G is k-word-representable is called the G's *representation number*.

The following observation follows trivially from the definitions.

Observation 1. *The class of complete graphs coincides with the class of 1-word-representable graphs. In particular, the complete graph's representation number is 1.*

A less trivial, but still simple fact mentioned in [6] is that the class of 2-word-representable graphs is precisely the class of circle graphs [4], which are defined by intersecting chords. Circle graphs were generalized to polygon–circle graphs [11], where edges are defined by intersecting inscribed k-gons for a fixed k. Note that except for the case of $k = 2$, such graphs are *not* the same as k-word-representable graphs. Indeed, in the case of k-word-representable graphs, $k \geq 3$, we have an edge xy if and only if x and y alternate, while in the case of polygon–circle graphs defined by intersecting inscribed k-gons, xy is an edge if and only if no cyclic shift of the subword induced by x and y, when reading the labels of the polygon corners around the circle in either direction, is $x \cdots xy \cdots y$.

1.1. Representation of crown graphs

A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets X and Y such that every edge connects a vertex in X to one in Y. A bipartite graph is *complete* if every vertex in X is connected to each vertex in Y. $K_{n,m}$ denotes the complete bipartite graph with the disjoint sets of sizes n and m, respectively. A *crown graph* (also known as a *cocktail party graph*) $H_{n,n}$ is a graph obtained from the complete bipartite graph $K_{n,n}$ by removing a perfect matching. Formally, $V(H_{n,n}) = \{1, \dots, n, 1', \dots, n'\}$ and $E(H_{n,n}) = \{ij' \mid i \neq j\}$. First four examples of such graphs are presented in Fig. 1.

Crown graphs are of special importance in the theory of word-representable graphs. More precisely, they appear in the construction of graphs requiring long words representing them [6]. Note that these graphs also appear in the theory of partially ordered sets as those defining partial orders that require many linear orders to be represented.

Each crown graph, being a bipartite graph, is a *comparability graph* (that is, a transitively orientable graph), and thus it can be represented by a concatenation of permutations [10]. Moreover, it follows from [6], and also is discussed in Section 7.4 in [8], that $H_{n,n}$ can be represented as a concatenation of n permutations but it cannot be represented as a concatenation of a fewer permutations. These results on crown graphs were obtained by exploiting the idea of representation of a poset as an intersection of several linear orders. Thus, the representation number of $H_{n,n}$ is at most n. See Table 1 (appearing in [8]) for the words representing the graphs in Fig. 1 as concatenation of permutations.

It was noticed in [7] that, for example, $H_{3,3}$ can be represented using two copies of each letter as $3'32'1'132'23'1'231'1$ (as opposed to three copies used in Table 1 to represent it) if we drop the requirement to represent crown graphs as concatenation of permutations. On the other hand, $H_{4,4}$ is the three-dimensional cube, which is the *prism graph* Pr_4, so that