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On the chromatic index of join graphs and triangle-free graphs with large maximum degree

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ABSTRACT

Deciding if the edges of a graph with n vertices and maximum degree Δ can be coloured with Δ colours is an NP-complete problem, but a conjecture proposed by Hilton and Chetwynd in 1984, along with a result of Niessen of 2001, suggests the existence of a linear-time algorithm for the problem when restricted to graphs with $\Delta \geqslant n/2$. Join graphs satisfy this restriction, but no polynomial-time algorithm for determining the chromatic index of a join graph is known. This paper presents some sufficient conditions for join graphs to have chromatic index equal to Δ and other results on the chromatic index of graphs with $\Delta \geqslant n/2$, proving that they all can be coloured with Δ colours when triangle-free.

1. Introduction

A *simple graph* G is an undirected graph without loops or multiple edges, whose vertex set is denoted by $V(G)$ and whose edge set is denoted by $E(G)$. The *Classification Problem* is the problem of determining the *chromatic index* of a given simple graph G, i.e. the minimum number of colours needed to colour all edges of G in a way that the same colour is not assigned to adjacent edges. Denoting the chromatic index and the maximum degree of G respectively by $\chi'(G)$ and $\Delta(G)$, Vizing's Theorem [24], which uses a recolouring procedure to colour the edges of any graph G with no more than $\Delta(G)+1$ colours, implies that we have either $\chi'(G) = \Delta(G)$, in which case G is said to be *Class* 1, or $\chi'(G) = \Delta(G)+1$, in which case G is said to be *Class* 2. Deciding whether a graph is *Class* 1 is an NP-complete problem [11], even when restricted, for example, to perfect graphs [1], to tripartite regular graphs [18] or to triangle-free cubic graphs [15]. Notwithstanding, the problem can be solved in polynomial time for some graph classes, such as bipartite [14] and complete multipartite graphs [10], graphs with a universal vertex [21] and graphs with acyclic core [8] (the *core* of a graph G, denoted by $\Lambda[G]$, is the subgraph induced by all vertices of degree Δ).

A graph G with n vertices and more than $\Delta(G)\lfloor n/2 \rfloor$ edges, possible only when n is odd, cannot be *Class* 1, since a colour can be assigned to at most $\lfloor n/2 \rfloor$ edges. Such a graph is said to be *overfull*. Furthermore, if G is not necessarily overfull, but has an overfull subgraph H with $\Delta(G) = \Delta(H)$, then G is also *Class* 2, and in this case we say that G is *subgraph-overfull* (shortly *SO*) and that H is a *Δ-overfull* subgraph of G. Not every *Class* 2 graph is *SO* (e.g. the Petersen graph, shown in Fig. 1), but, for graphs with large maximum degree, this equivalence is suspected to be true.

In 1984, Hilton and Chetwynd proposed the following conjecture:

Hilton and Chetwynd's conjecture (*[2]*). A graph G with n vertices and maximum degree $\Delta \geqslant n/2$ is *Class* 2 if and only if it is *SO*.

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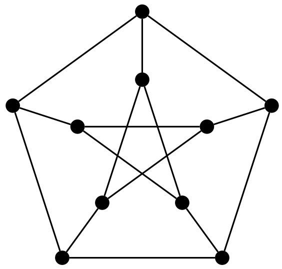

Fig. 1. The Petersen graph.

This conjecture was later replaced by the *Overfull Conjecture* [3,9], which covers the former and states that the equivalence holds for graphs with $\Delta > n/3$. The restriction seems to be the best possible, since P^*, the *Class* 2 graph obtained from the Petersen graph by removing any vertex, is not *SO* and has $\Delta = n/3$. Moreover, Niessen showed in 1994 [19] that a graph with $\Delta \geqslant n/2$ has at most one Δ-overfull induced subgraph, which can be found in polynomial time. In 2001 [20], Niessen also showed that a graph with $\Delta > n/3$ has at most three Δ-overfull induced subgraphs, which can be found in polynomial time by the *Algorithm* 2 of the paper. In turn, the *Algorithm* 1 of the paper is an improvement of the algorithm of 1994 which can run in linear time.

The *join* operation of two graphs G_1 and G_2, denoted by $G_1 * G_2$, is the graph defined by $V(G_1 * G_2) = V(G_1) \cup V(G_2)$ and

$$E(G_1 * G_2) = E(G_1) \cup E(G_2) \cup \{uv\colon u \in V(G_1) \text{ and } v \in V(G_2)\}.$$

A graph G is said to be a *join graph* if there are two graphs G_1 and G_2 such that $G = G_1 * G_2$. If G_1 and G_2 are not the same K_1 graph, we can assume without loss of generality that $V(G_1)$ and $V(G_2)$ are disjoint, because, if they are not, we can simply replace G_1 and G_2 respectively by the graphs G_1' and G_2' defined by:

$$V(G_1') = V(G_1); \qquad E(G_1') = E(G_1) \cup \{uv\colon u, v \in V(G_1) \cap V(G_2)\};$$
$$V(G_2') = V(G_2) \setminus V(G_1); \qquad E(G_2') = E(G_2) \setminus \{uv \in E(G_2)\colon u \in V(G_1)\}.$$

As a graph is a *join graph* if and only if it is a K_1 or its *complement* is disconnected, join graphs can be recognised in linear time with the algorithms presented in [12]. Since it can be easily shown that a join graph G has no more than $2\Delta(G)$ vertices, Hilton and Chetwynd's Conjecture suggests, along with Algorithm 1 of [20], that the chromatic index of a join graph can be decided in linear time. However, only partial results are known. Assuming without loss of generality that $n_1 = |V(G_1)| \leqslant |V(G_2)| = n_2$, some sufficient conditions for $G = G_1 * G_2$ to be *Class* 1 are:

(1) $\Delta(G_1) > \Delta(G_2)$ [22];
(2) $\Delta(G_1) = \Delta(G_2) = \Delta$ and:
  (2.1) both G_1 and G_2 are *Class* 1 [22];
  (2.2) G is regular [22];
  (2.3) both G_1 and G_2 are disjoint unions of cliques [22];
  (2.4) $n_1 = n_2$ and each connected component of $G_1 \cup G_2$ has at most $\Delta + 1$ vertices [16];
  (2.5) $n_1 = n_2$, each connected component of G_1 has at most $\Delta + 1$ vertices, G_2 is bipartite and either Δ is odd or G_2 has no $K_{\Delta,\Delta}$ as induced subgraph [16];
  (2.6) $\Lambda[G_1]$ is edgeless [16];
  (2.7) G_1 is a cograph (i.e. graph with no P_4 as an induced subgraph), $|V(\Lambda[G_1])| \leqslant n_2 - |V(\Lambda[G_2])|$, and $\Lambda[G_1]$ is acyclic [16];
(3) $\Delta(G_1) < \Delta(G_2)$ and $n_1 = n_2$ [22];
(4) $\Delta(G_2) < n_2 - n_1$ [17];
(5) G_1 and G_2 are regular and $n_1 + n_2$ is even [23].

In edge-colouring, our interest lies mainly on connected graphs, since the chromatic index of a disconnected graph is the maximum among the chromatic indices of its connected components, and on graphs in which every vertex is a Δ-vertex (a vertex of degree Δ) or adjacent to a Δ-vertex, since the removal of a non-Δ-vertex non-adjacent to a Δ-vertex does not change the chromatic index of the graph [17]. An important subclass of join graphs is the class of the connected cographs. *Almost every* graph can be turned into a cograph with the addition of *a few* edges [6], and many NP-hard graph problems, including vertex-colouring, have linear-time solutions when restricted to cographs [4,5], but no polynomial-time algorithm is known for deciding the chromatic index of a cograph. Two remarkable subclasses of connected cographs in which every non-overfull graph is *Class* 1 are the class of the complete multipartite graphs [10] and the class of the connected quasi-thresholds graphs [21], which are a special case of graphs with a universal vertex [13].

This paper is organised as follows: in Section 2, we present further partial results on the chromatic index of join graphs, as well as a partial proof and a generalisation of a conjecture proposed in [16], about the case wherein the graph with minimum order involved in the join operation is a cograph with acyclic core; in Section 3, we restate Hilton and Chetwynd's Conjecture in terms of Algorithm 1 of [20] and prove that all triangle-free graphs with $\Delta \geqslant n/2$ are *Class* 1 by revisiting Vizing's recolouring procedure.

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