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Reliability analysis of Cayley graphs generated by transpositions

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ABSTRACT

Let Γ_n be the symmetric group on $\{1, 2, \ldots, n\}$ and S be the generating set of Γ_n. The corresponding Cayley graph is denoted by $\Gamma_n(S)$. If all elements of S are transpositions, a simple way to depict S is via a graph, called the *transposition generating graph* of S, denoted by $A(S)$ (or say simply A), where the vertex set of A is $\{1, 2, \ldots, n\}$, there is an edge in A between i and j if and only if the transposition $(ij) \in S$, and $\Gamma_n(S)$ is called a *Cayley graph obtained from a transposition generating graph A*. In this paper, by exploring and utilizing the structural properties of these Cayley graphs, we obtain that the pessimistic diagnosability of $\Gamma_n(S)$ is equal to $2|E(A)| - 2$ if A has no triangles or $2|E(A)| - 3$ if A has a triangle. As corollaries, the pessimistic diagnosability of many kinds of graphs such as Cayley graphs generated by unicyclic graphs, wheel graphs, complete graphs, and tree graphs is obtained.

1. Introduction

With the rapid development of technology, a multiprocessor system may contain thousands of processors. As a significant increase in the number of processors, some processors may fail in the system, so processor fault identification plays an important role for reliable computing. A system (network) can be represented by a graph, where each vertex represents a processor, and an edge indicates that the communication link between processors.

The process of identifying faulty processors in a system by conducting tests on various processors and interpreting the test outcomes is known as the *system-level diagnosis*. For the purpose of self-diagnosis of a system, a number of models have been proposed. In particular, the PMC model, first proposed by Preparata et al. [18], is well known and widely used. In this model, the diagnosis of the system is achieved through two linked processors testing each other. This evaluation is reliable if and only if the testing processor is fault-free.

For a system, its diagnostic capability can be measured by its *precise one-step diagnosability* [18], namely the largest integer t such that for any fault set F of faults with $|F| \leq t$, the precise one-step diagnosis can be achieved. Under the precise strategy, fault-set F identified only contains all faulty vertices. But if the number of faulty processors exceeds the precise one-step diagnosability of the system, the precise one-step diagnosis fails.

To overcome the previously mentioned limitation of the precise one-step diagnosis, Kavianpour and Friedman [13,14] investigated an alternative approach to system-level diagnosis by lowering the diagnosis demand to the extent that the processors are identified completely but not necessarily correctly. This approach is known as the *pessimistic one-step diagnosis*. Using the pessimistic strategy, fault-set F identified is allowed to contain all faulty vertices and some fault-free vertices. Chwa and Hakimi [8] characterized the diagnosable systems, and Sullivan [19] devised a polynomial time algorithm for determining the diagnosability of a system. The pessimistic diagnosis strategy is a classic strategy based on the PMC

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model [13]. In this strategy, all faulty processors to be isolated within a set having at most one fault-free processor. The pessimistic (one-step) diagnosability measure of the diagnostic capability of a graph is defined as follows:

Definition 1. A system is t/t-*diagnosable* if, provided the number of faulty processors is bounded by t, all faulty processors can be isolated within a set of size at most t with at most one fault-free processor mistaken as a faulty one. The *pessimistic diagnosability* of a system G, denoted by $t_p(G)$, is the maximal number of faulty processors so that the system G is t/t-diagnosable.

Compared with the precise one-step diagnosability, the pessimistic one-step diagnosability of a system is significantly increased at the expense of a small number of fault-free processors being replaced. Kavianpour and Kim [15] showed that the hypercube is $(2n-2)/(2n-2)$-diagnosable. The enhanced hypercube has been shown to be $2n/2n$-diagnosable by Wang [24]. Fan [9] showed that the degree of diagnosability of the Möbius cube using the pessimistic strategy is $2n-2$. Wang et al. [25] studied the pessimistic diagnosability of the k-ary n-cubes. The pessimistic diagnosability of the alternating group graph AG_n and the hypercube-like network (BC network) were obtained by Tsai in [21] and [20], respectively. Hao et al. [12] gave the pessimistic diagnosabilities of some general regular graphs. Recently, the pessimistic diagnosability of the (n, k)-arrangement graph, (n, k)-star graph and the balanced hypercube, the bubble-sort star graphs and augmented k-ary n-cubes were determined in [10] and [11], respectively.

Network topology is an important factor because it affects the performance of the network. Cayley graphs have been recognized as topologies of multiprocessor systems [1,16]. The star graphs and bubble-sort graphs are Cayley graphs on the symmetric groups with the set of *transpositions* as the generator sets. Some results about topological properties and routing problems on Cayley graphs generated by transpositions can be found in [2,4–7,16,17,23,27,28], etc. Transpositions play a very important role in investigating the properties of these graphs.

In this paper, we study the Cayley graphs generated by transpositions. By exploring and utilizing the structural properties of the Cayley graphs generated by m transpositions, we obtain the following: (1) If the transposition generating graph A has no triangles, the pessimistic diagnosability is equal to $2m-2$. (2) If the transposition generating graph A has a triangle, the pessimistic diagnosability is equal to $2m-3$. As corollaries, the pessimistic diagnosability of Cayley graphs generated by unicyclic graphs, wheel graphs, complete graphs, tree graphs, etc. is obtained.

The remainder of this paper is organized as follows. Section 2 introduces preliminaries and the structure properties of the Cayley graphs generated by m transpositions and gives some properties. We prove our main results in Section 3, and add concluding remarks in the last section.

2. Preliminaries

In this section, we give some terminologies and notations. For notation and terminology not defined here we follow [3] and [26]. We use a graph, to represent a system (network). Throughout this paper, all graphs are finite, undirected without loops.

Let G be a simple undirected graph. Let $|V(G)|$ be the size of vertex set and $|E(G)|$ be the size of edge set. Two vertices u and v are *adjacent* if $(u, v) \in E(G)$, the vertex u is called a neighbor of v, and vice versa. For a vertex $u \in V(G)$, let $N_G(u)$ denote a set of vertices in G adjacent to u. $|N_G(u)|$ is the *degree* of u. Let $\Delta(G)$(resp. $\delta(G)$) refer to the *maximum (resp. minimum) degree* of vertices in G. For a vertex set $U \subseteq V(G)$, the *neighborhood* of U in G is defined as $N_G(U) = \bigcup_{v\in U} N_G(v) - U$. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *components* of a graph G are its maximally connected subgraphs. A component is *trivial* if it has only one vertex; otherwise, it is *nontrivial*. The *connectivity* of a graph G, denoted by $\kappa(G)$, is defined as the minimum number of vertices whose removal results in a disconnected or trivial graph. If $|N_G(u)| = k$ for any vertex in G, then G is k-*regular*. Let G be a connected graph, if $G - S$ is still connected for any $S \subseteq V(G)$ with $|S| \le k-1$, then G is k-*connected*. A k-regular graph is *maximally connected* if it is k-connected. A vertex $v \in V(G)$ is a *cut-vertex* if $G - \{v\}$ has more connected components than G has.

Let Γ be a finite group, and let S be a set of elements of Γ such that the identity of the group does not belong to S. The *Cayley graph* $\Gamma(S)$ is the direct graph with vertex set Γ, such that there is an arc from u to v if and only if there is an $s \in S$ such that $u = vs$. Assume that S does not contain the identity of the group Γ to ensure that $\Gamma(S)$ is simple. If $S = S^{-1}$, then $\Gamma(S)$ can be viewed as an undirected graph. The Cayley graph $\Gamma(S)$ is connected if and only if S is a generating set for Γ. A Cayley graph is vertex transitive.

In this paper, we choose the finite group to be Γ_n, the symmetric group on $\{1, 2, \ldots, n\}$ and the generating set S to be a set of transpositions. The corresponding Cayley graph is denoted by $\Gamma_n(S)$. Note that S has only transpositions, there is an arc from vertex u to vertex v if and only if there is an arc from v to u. So we regard these Cayley graphs as undirected graphs by replacing each pair of arcs between two vertices by an edge. With transpositions as the generating set, a simple way to depict S is via a graph with vertex set $\{1, 2, \ldots, n\}$, there is an edge between i and j if and only if the transposition $(ij) \in S$. This graph is called the *transposition generating graph* of S, denoted by $A(S)$ (or A, if there is no ambiguous). $\Gamma_n(S)$ is also called a *Cayley graph obtained from a transposition generating graph* $A(S)$. Throughout this paper, we require $A(S)$ is connected to ensure that the Cayley graph generated by transpositions is connected.

Cheng et al. [6] obtained the following properties.