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Weighted antimagic labeling[☆,☆☆]

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ABSTRACT

A graph $G = (V, E)$ is *weighted-k-antimagic* if for each $w : V \to \mathbb{R}$, there is an injective function $f : E \to \{1, \dots, |E| + k\}$ such that the following sums are all distinct: for each vertex u, $\sum_{v:uv\in E} f(uv) + w(u)$. When such a function f exists, it is called a (w, k)-*antimagic labeling* of G. A connected graph G is *antimagic* if it has a $(w_0, 0)$-antimagic labeling, for $w_0(u) = 0$, for each $u \in V$.

In this work, we prove that all the complete bipartite graphs $K_{p,q}$, are weighted-0-antimagic when $2 \le p \le q$ and $q \ge 3$. Moreover, an algorithm is proposed that computes in polynomial time a $(w, 0)$-antimagic labeling of the graph. Our result implies that if H is a complete partite graph, with $H \ne K_{1,q}, K_{2,2}$, then any connected graph G containing H as a spanning subgraph is antimagic.

1. Introduction

A connected graph $G = (V, E)$ with m edges and n vertices is *antimagic* if there exists a bijective function $f : E \to \{1, \dots, m\}$, such that the following sums are all different: for each vertex u, $\sum_{e\in E(u)} f(e)$, where $E(u)$ is the set of edges incident to vertex u. Hartsfield and Ringel conjectured that every connected graph with at least two edges is antimagic [7].

It is easy to see that a graph with n vertices and maximum degree $n - 1$ is antimagic. It is also easy to see that cycles and paths are antimagic. Less obvious, in [2], it was proved that the class of antimagic graphs contains every complete partite graph, except K_2, and every graph with n vertices and maximum degree $n - 2$. This latter result was improved in [14], where it was proved that every graph with maximum degree at least $n - 3$ is antimagic as well. Cartesian products of various graphs, as path and cycles, have also been proved to be antimagic [4,11,12]. More general, in [2], it was also proved that there is a constant c such that any graph with n vertices and minimum degree at least $c \log n$ is antimagic. This result is obtained by applying Lovász's Local Lemma and, in fact, shows the existence of a much more general kind of labeling which we discuss later. In [5], it was proved that regular bipartite graphs are antimagic. This result was extended in [6] to regular graphs of odd degree, and recently, proved for all regular graphs [3]. However, the conjecture is still open even for trees, where the best result, proved in [10], shows that trees with at most one vertex of degree two are antimagic.

In order to gain more intuition about the conjecture, it is natural to explore some variations. Among the ideas considered so far, in [8], the following notion was considered. Given an integer k, a graph $G = (V, E)$ is *weighted-k-antimagic* if for any weight function $w : V \to \mathbb{R}$, there is an injective function $f : E \to \{1, \dots, |E| + k\}$, such that the following sums are all

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different: for each vertex u, $w(u)+\sum_{e\in E(u)} f(e)$. Such a function f is called a (w,k)-*antimagic labeling*. Clearly, if a graph is weighted-0-antimagic, then it is antimagic.

The proof of Theorem 2 in [2] actually shows that there is a constant c such that every graph with n vertices and minimum degree at least $c\log n$ is weighted-0-antimagic. Besides this latter result only a few more families are known to be weighted-0-antimagic. In [8], by using the Combinatorial Nullstellensatz Theorem [1], it was proved that any graph on n vertices having a 2-factor with each cycle of length 3, is weighted-0-antimagic, if $n=3^l$, for some integer l. Later, in [9], this result was extended to any odd prime instead of 3.

On the other hand, not every graph is weighted-0-antimagic, as one can easily see, by considering the complete bipartite graph $K_{1,n-1}$ [13]. Then, it is natural to ask if there is a constant k such that every graph is weighted-k-antimagic. A partial answer to this question was given in [13] where it was shown that $K_{1,n-1}$ is weighted-2-antimagic, and it is weighted-1-antimagic, when n is odd. Moreover, it was also proved that a path on n vertices, with n prime, is weighted-1-antimagic.

The main characteristic of all the proofs, of the previously mentioned results, is that they are non-constructive as they are based on the Combinatorial Nullstellensatz Theorem [1]. For instance, for each weight function w there is a $(w,1)$-antimagic labeling of $K_{1,12}$ and we do not know how to construct it.

Our contribution

The purpose of this work is to present an algorithmic approach to deal with weighted-k-antimagic graphs. This approach is new in this context and allows to generalize some known results, together with providing an explicit construction of the labelings. Even though some proofs appearing in the previously mentioned works on antimagic labeling are constructive, it is not clear how the associated algorithms can be transferred to the context of weighted-k-antimagic labeling. For instance, from the proof of Theorem 1.3 in [2], it is possible to deduce the existence of a procedure to construct a $(w,0)$-antimagic labeling of $K_{p,q}$, but only for some specific w, those which are zero in the vertices of the smaller independent set.

In Section 2, we prove that there exists an algorithm for arbitrary weights.

Result 1. *For each $2\le p\le q$ and $q\ge 3$, the graph $K_{p,q}$ is weighted-0-antimagic. Moreover, there exists an algorithm which on input $w\in\mathbb{R}^{p+q}$ runs in polynomial time and returns a $(w,0)$-antimagic labeling of $K_{p,q}$.*

Result 1 is tight. In [13], it was noticed that $K_{1,n-1}$ is not weighted-0-antimagic. On the other hand, $K_{2,2}$ is not weighted-0-antimagic either. In fact, let $\{x_1,x_2\}$ and $\{y_1,y_2\}$ be the independent sets of $K_{2,2}$. Let w be the weight function given by $w(x_1)=w(x_2)=0$ and $w(y_1)=w(y_2)=1$. For the sake of contradiction, let us assume that $f:\{x_1y_1,x_2y_2,x_2y_1,x_2y_2\}\to\{1,2,3,4\}$ is a $(w,0)$-antimagic labeling of $K_{2,2}$. W.l.o.g., we can assume that $f(x_1y_1)=1$. Hence, since $1+4=2+3$, we are forced to set $f(x_2y_2)=4$. When $f(x_1y_2)=3$ the vertex sum at vertex x_1 is $0+1+3=4$ and the sum at vertex y_1 is $1+1+2=4$. Otherwise, when $f(x_1y_2)=2$, the sum at vertex x_2 is $0+3+4=7$ and the sum at vertex y_2 is $1+2+4=7$. Therefore, there is no $(w,0)$-labeling of $K_{2,2}$.

From the graph theoretical point of view, the correctness of our algorithm implies that every complete bipartite graph with the adequate size is weighted-0-antimagic. To the best of our knowledge, this result is new.

From this, it follows that a large class of graphs are weighted-0-antimagic, thus antimagic. In fact, in Section 3, we prove the following result which is a generalization of Theorem 1.3 in [2] from antimagic labeling to weighted-0-antimagic labeling.

Result 2. *Let H be an arbitrary complete partite graph with at least five vertices and $H\ne K_{1,n-1}$. Then, any graph containing H as a spanning subgraph is weighted-0-antimagic. Moreover, given a weight function w, a $(w,0)$-antimagic labeling can be computed in polynomial time.*

The ideas used in our algorithm allow us to give a short algorithmic proof of a generalization of the previously cited result about graphs having universal vertices [13]. More precisely, our result clarifies how far from being weighted-1-antimagic the graph $K_{1,n-1}$ is, by giving a complete characterization of those weights w for which a $(w,0)$-antimagic labeling of $K_{1,n-1}$ does not exist. This information allows us to prove that the only graph with a universal vertex which is not weighted-1-antimagic is the graph $K_{1,n-1}$, when n is even.

Result 3. *Each graph G on n vertices having a universal vertex is weighted-1-antimagic, unless $G=K_{1,n-1}$ and n is even. Given a weight function w, a $(w,1)$-antimagic labeling can be constructed in polynomial time.*

Surprisingly, despite its simplicity, our technique easily works if we replace the set $\{1,\dots,m\}$ by any set of m consecutive positive integers.

We can further extend previous result to any set of given integers. The proof of this result requires to enhance our algorithmic techniques to consider new difficulties not appearing in the current setting. We present this result in a forthcoming paper.

2. Antimagic algorithm

In this section, we present an algorithm, called Antimagic, which receives a weight function w and construct a $(w,0)$-antimagic labeling of $K_{p,q}$.