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Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Planar graphs without chordal 6-cycles are 4-choosable

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ARTICLE INFO

Article history:
Received 31 May 2017
Received in revised form 21 December 2017
Accepted 7 March 2018
Available online xxxx

Keywords:
List coloring
Planar graph
Cycle
Chord

ABSTRACT

A graph G is k-choosable if it can be colored whenever every vertex has a list of at least k available colors. In this paper, we prove that every planar graph without chordal 6-cycles is 4-choosable. This extends a known result that every planar graph without 6-cycles is 4-choosable.

1. Introduction

All graphs considered in this paper are simple, finite and undirected, and we follow [2] for the terminologies and notation not defined here. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For a vertex $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to v, and let $d(v) = |N(v)|$ denote the degree of v. A *k-vertex*, a *k^+-vertex* or a *k^--vertex* is a vertex of degree k, at least k, or at most k, respectively. We use $\Delta(G)$ and $\delta(G)$ (or simply Δ and δ) to denote the maximum degree and the minimum degree of G, respectively. A k-cycle is a cycle of length k, and a 3-cycle is usually called a *triangle*. Two cycles are *adjacent* (or *intersecting*) if they share at least one edge (or vertex, respectively). Given a cycle C of length k in G, an edge $xy \in E(G)\backslash E(C)$ is called a *chord* of C if $x, y \in V(C)$. Such a cycle C is also called a chordal k-cycle.

A proper coloring of a graph G is a mapping ϕ from $V(G)$ to the color set $[k] = \{1, 2, \dots, k\}$ such that $\phi(x) \neq \phi(y)$ for every two adjacent vertices x and y of G. We say that L is an assignment for the graph G if it assigns a list $L(v)$ of possible colors to each vertex v of G. If G has a proper coloring ϕ such that $\phi(v) \in L(v)$ for all vertices v, then we say that G is L-colorable or ϕ is an L-coloring of G. The graph G is *k-choosable* if it is L-colorable for every assignment L satisfying $|L(v)| \geq k$ for any vertex v. The *choice number* or *list chromatic number* $\chi_l(G)$ of G is the smallest k such that G is k-choosable.

The concept of list coloring of a graph was introduced by Vizing [10] and Erdős, Rubin and Taylor [3], respectively. Thomassen [9] showed that every planar graph is 5-choosable. Examples of plane graphs which are not 4-choosable and plane graphs of girth 4 which are not 3-choosable were given by Voigt [11,12]. Since every planar graph without 3-cycles is 3-degenerate and hence is 4-choosable. Wang and Lih [15] showed that planar graphs without intersecting 3-cycles are 4-choosable. Further, it was proved that every k-cycle-free planar graph is 4-choosable for $k = 4$ in [8], for $k = 5$ in [7,14], for $k = 6$ in [5,7,13], and for $k = 7$ in [4]. In 2002, Wang and Lih [15] raised the following conjecture:

Conjecture 1. *Every planar graph without adjacent 3-cycles is 4-choosable.*

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https://doi.org/10.1016/j.dam.2018.03.014

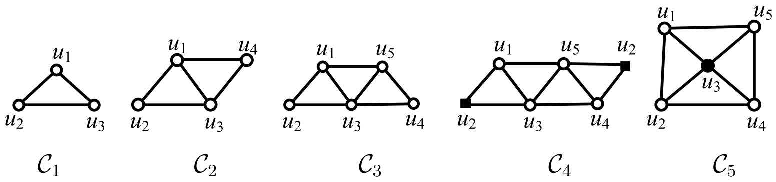

Fig. 1. All possible clusters in G.

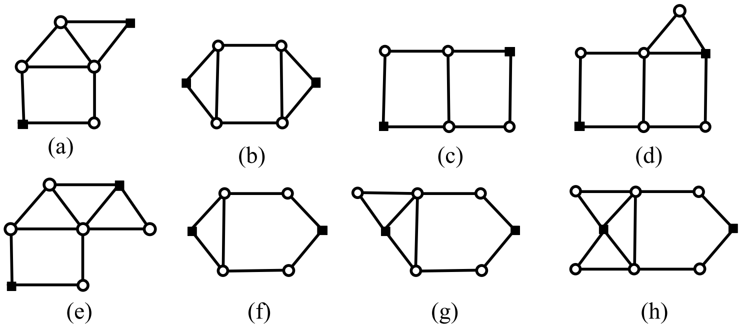

Fig. 2. The configurations in Lemma 4.

Equivalently, Conjecture 1 states that planar graphs without chordal 4-cycles are 4-choosable. So far Conjecture 1 has remained to be open. However, in this paper, we shall prove the following related result:

Theorem 1. *Planar graphs without chordal* 6*-cycles are* 4*-choosable.*

Clearly, Theorem 1 is an extension to the result in [5,7,13].

2. Proof

This section is devoted to show Theorem 1. Let G be a plane graph. For $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [v_1v_2 \cdots v_n]$ if $v_1, v_2, \ldots, v_n$ are the boundary vertices of f in a cyclic order. For a face $f \in F(G)$, let $d(f)$ denote the degree of f, i.e., the number of edges in $b(f)$. If a face is of degree k, at least k, or at most k, we call it a *k-face*, *k^+-face*, or *k^--face*. Given a vertex $v \in V(G)$, let $T(v)$ denote the set of 3-faces incident with v, and let $t(v) = |T(v)|$. For a face $f \in F(G)$, let $m_3(f)$ denote the number of 3-faces adjacent to f. A *cluster* $\mathcal{C}$ is a connected subgraph of G consisting of a set of 3-faces such that any 3-face not in $\mathcal{C}$ is not adjacent to any 3-face in $\mathcal{C}$. We say that a face f is *adjacent* to a cluster $\mathcal{C}$ if f is adjacent to a 3-face in $\mathcal{C}$.

Suppose, to the contrary, that Theorem 1 is false. Let G be a counterexample to Theorem 1 with fewest vertices. Namely, G is a planar graph without chordal 6-cycles that is not 4-choosable, but $G - v$ is 4-choosable for any vertex $v \in V(G)$. Obviously, G is connected. Embed G into the plane.

We investigate the structural properties of G first. The following lemma holds trivially.

Lemma 2. $\delta(G) \geq 4$.

The literature [6] gives all possible clusters in G and describes certain small faces that can be adjacent to a given cluster.

Lemma 3 (*[6]*)**.** *There are only five possible clusters of* 3*-faces in G, depicted in Fig. 1.*

In Fig. 1 (and also in Fig. 2), solid squares denote two copies of the same vertex, e.g., $\mathcal{C}_4$ has only five distinct vertices. Moreover, let $\mathcal{C}_3^*$ and $\mathcal{C}_5^*$ denote, respectively, two special clusters $\mathcal{C}_3$ and $\mathcal{C}_5$ in Fig. 1 where $d(u_1) = d(u_3) = d(u_5) = 4$.

Lemma 4 (*[6]*)**.** *G satisfies the following statements* (1)–(9)

(1) *$\mathcal{C}_2$ is adjacent to at most one* 4*-face forcing an identification as shown in* Fig. 2(a).

(2) *If a* 4*-face is adjacent to two* 3*-faces, then they must be as shown in* Fig. 2(b).

(3) *Two adjacent* 4*-faces force an identification as in* Fig. 2(c), *and there is only one way for them to be adjacent to a* 3*-face as in* Fig. 2(d).

(4) *$\mathcal{C}_3$ is adjacent to a* 4*-face in a unique way, as shown in* Fig. 2(e).

Please cite this article in press as: D.-Q. Hu, et al., Planar graphs without chordal 6-cycles are 4-choosable, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2018.03.014.