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Complexity of minimum irreducible infeasible subsystem covers for flow networks

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ABSTRACT

For an infeasible network flow system with supplies and demands, we consider the problem of finding a minimum irreducible infeasible subsystem cover, i.e., a smallest set of constraints that must be dropped to obtain a feasible system. The special cases of covers which only contain flow balance constraints (node cover) or only flow bounds (arc cover) are investigated as well. We show strong $\mathcal{NP}$-hardness of all three variants. Furthermore, we show that finding minimum arc covers for assignment problems is still hard and as hard to approximate as the set covering problem. However, the minimum arc cover problem is polynomially solvable for networks on cactus graphs. This leads to the development of two different fixed parameter algorithms with respect to the number of elementary cycles connected at arcs and the treewidth, respectively. The latter can be adapted for node covers and the general case.

1. Introduction

Analyzing infeasibility of linear programs (LPs) is an important topic, since it can help to find disrupted data or locate modeling errors. One tool for this purpose is small sets of constraints whose removal renders the LP feasible. This requires to remove at least one constraint from every *irreducible infeasible subsystem* (IIS), i.e., an infeasible subsystem such that each proper subsystem is feasible. Thus, we are interested in *minimum IIS covers* (minIISCs).

In this paper, we are concerned with the special case of network flow systems

$$x(\delta^+(v)) - x(\delta^-(v)) = b(v) \qquad \forall\, v \in V, \tag{1a}$$

$$\ell \le x \le u, \tag{1b}$$

for a simple, directed graph $G = (V, A)$ with upper bounds $u \in \mathbb{R}^A$, lower bounds $\ell \in \mathbb{R}^A$, and a supply vector $b \in \mathbb{R}^V$. For $S \subseteq V$ and $\bar{S} := V \setminus S$, we use the following standard notation: $\delta^+(S) := \{(v, w) \in A \mid v \in S,\ \ w \in \bar{S}\}$, $\delta^-(S) := \{(v, w) \in A \mid v \in \bar{S},\ \ w \in S\}$, and $\delta(S) := \delta^+(S) \cup \delta^-(S)$. Moreover, for a finite index set I, a vector $y \in \mathbb{R}^I$, and $I' \subseteq I$, we write $y(I') := \sum_{i \in I'} y_i$ and use $y(i) := y(\{i\}) = y_i$. The number of nodes and arcs of G are $n := |V|$ and $m := |A|$, respectively.

In order to avoid trivial infeasibilities, we assume that $\ell \le u$. Furthermore, w.l.o.g. $u \ge 0$ throughout the article. In later parts, we will also assume that the supply/demand is *balanced*, i.e., $b(V) = 0$.

The analysis of IISs and IIS covers for LPs has been treated extensively in the literature (see Section 1.1 for a survey). However, to the best of our knowledge, the special case of IIS covers for flow networks has not been treated so far. In this article we investigate three kinds of such IIS covers: An *IIS node cover* (INC) covers all IISs of the network problem by

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node constraints alone, i.e., flow conservation equations (1a). Similarly, an *IIS arc cover* (IAC) contains only arc constraints, i.e., lower or upper bounds (1b). The combination of both yields general IIS covers. The corresponding minimization problems are then called MINC, MIAC, and MIC, respectively.

The goal of this article is to extend the knowledge on finding minimum IIS covers by considering the base case of flow networks. We investigate the structural properties of the three types of IIS covers and, in particular, investigate their computational complexity and (non-)approximability properties.

To this end, we first establish $\mathcal{NP}$-hardness of the three problems MIC, MIAC, and MINC in Section 2. In Section 3, we study characteristics of IIS covers and MIAC, in particular. For instance, we show that MIAC is approximable within $c\,(n-1)$ for every constant $c > 0$. We demonstrate that MIC can be formulated as a MIAC instance; hence, MIC can also be approximated within $c\,n$. Furthermore, we introduce the concept of redundancy with respect to covering of IISs. This leads to two preprocessing rules that allow to simplify the network. We also examine the relation between MIAC and MINC. We then show in Section 4 that MIAC is $\mathcal{NP}$-hard to approximate within $c \ln n$, even on assignment problems. Moreover, we develop polynomial time algorithms on trees, cycles, and more generally on cactus graphs. Furthermore, we give two fixed parameter algorithms with respect to the number of elementary cycles connected at arcs (MIAC) and with respect to the treewidth (all three variants).

Thus, on the one hand it turns out that minimum IIS covers are already hard to compute for flow networks. In fact, MIAC is as hard to approximate as the minimum IIS cover problem for general LPs. On the other hand, one can use the underlying graph structure to derive tractable special cases. Consequently, this article complements the existing knowledge on computing IIS covers for infeasible LPs to a certain extent, which we review in the next subsection.

1.1. Literature overview

The analysis of general infeasible linear systems has been extensively investigated. For a broad overview, we refer to the book by Chinneck [12] (and references therein). Moreover, we also mention the article of Greenberg [22], which gives a unified presentation of infeasibility and redundancy. In [20,21], he also studied infeasible networks and gave heuristics to "localize" the cause of infeasibility. In the following, we concentrate on minIISC results.

For general LPs, minIISC is equivalent to its complementary problem, the *maximum feasible subsystem* (maxFS) problem. Chakravarti [9] showed that maxFS is strongly $\mathcal{NP}$-hard (thus, minIISC is also strongly $\mathcal{NP}$-hard). While minIISC and maxFS are equivalent with respect to optimality, this does not hold for approximability, see Amaldi and Kann [4,5].

MinIISC can be formulated as a hitting set (or as a set covering) problem, where the sets are all IISs I of a linear system with p constraints. We can hence solve minIISC optimally via the following integer program (IP):

$$\begin{aligned} \min\ & \mathbb{1}^{\top} y \\ \text{s.t.}\ & \sum_{i \in \mathcal{I}} y_i \geq 1 \quad \forall\, \mathcal{I} \in I \\ & y \in \{0,1\}^p, \end{aligned} \tag{2}$$

where we identify the constraints by their indices $1, \ldots, p$, and $\mathbb{1}$ denotes the all-ones vector of appropriate dimension.

Parker and Ryan [31] present an iterative process to solve (2), in which IISs are generated dynamically. Here, (2) is solved for a partial set of IISs. An integral solution can be used to efficiently find uncovered IISs by using a result of Gleeson and Ryan [17]: The index sets of IISs of an infeasible linear system are exactly the supports of the vertices of the associated alternative polyhedron. If an uncovered IIS is found, it is added to the set, and the process is iterated. A branch-and-cut approach to solve (2) is given in [32], which generates IISs on the fly. Chinneck presented heuristics to solve minIISC [10] and maxFS [11].

Sankaran [33] proved $\mathcal{NP}$-hardness of minIISC for $Dy \leq d$ with a transposed node–arc-incidence matrix D. Hence, his result does not carry over to our case. He also presented an easy special case: If the concatenated matrix $[D\,d]$ is totally unimodular, minIISC can be solved in polynomial time for the general linear system $Dy \leq d$.

Furthermore, Amaldi and Kann [5] use the name "unsatisfied linear relations" (MIN ULR) for MinIISC. They discussed the (non-)approximability properties of $Dy\mathcal{R}d$ for $\mathcal{R} \in \{\neq, =, \geq, >\}$ and arbitrary D and d. These results can be extended to the constrained C MIN ULR, where some constraints are mandatory and have to be satisfied. For $\mathcal{R} \in \{=, \geq, >\}$, they showed that MIN ULR can be approximated within $m + 1$, where m is the number of variables, by the following observation: In the approach by Parker and Ryan mentioned above, including every constraint of a newly found IIS in the IIS cover would increase the cover by at most $m+1$ instead of 1, since each IIS can have at most $m+1$ constraints, see Motzkin [30]. Amaldi and Kann [5] also claimed (non-) approximability results for different versions of MIN ULR on node–arc-incidence matrices using Sankaran's results. However, these results are incorrect, since Sankaran [33] used *arc–node*-matrices.

In the following, we write (1) as $Mx = b$, $\ell \leq x \leq u$, where the matrix M is the totally unimodular node–arc-incidence matrix of G. In this context, McCormick [28] considered the following slack-formulation for an infeasible network flow:

$$\begin{aligned} \min\ & \|s\| \\ & Mx = b + s_1 \\ & \ell - s_2 \leq x \leq u + s_3 \\ & s_2, s_3 \geq 0, \end{aligned} \tag{3}$$