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A linear complementarity based characterization of the weighted independence number and the independent domination number in graphs

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ABSTRACT

The linear complementarity problem is a continuous optimization problem that generalizes convex quadratic programming, characterization of Nash equilibria of bimatrix games and several such problems. This paper presents a continuous optimization formulation for the weighted independence number of a graph by characterizing it as the maximum weighted ℓ_1 norm over the solution set of a linear complementarity problem (LCP). The minimum ℓ_1 norm of solutions of this LCP forms a lower bound on the independent domination number of the graph. Unlike the case of the maximum ℓ_1 norm, this lower bound is in general weak, but we show it to be tight if the graph is a forest. Using methods from the theory of LCPs, we then obtain a stronger variant of the Lovász theta and a new sufficient condition for a graph to be well-covered, i.e., for all maximal independent sets to also be maximum. This latter condition is also shown to be necessary for well-coveredness of forests. Finally, the reduction of the maximum independent set problem to a linear program with (linear) complementarity constraints (LPCC) shows that LPCCs are hard to approximate.

1. Introduction

An undirected graph G is given by the pair (V, E) where V is a finite set of *vertices* and E is a set of unordered pairs of vertices called *edges*. Two vertices $i, j \in V$ are said to be adjacent if there exists an edge $(i, j) \in E$ between them. Adjacent vertices are also called neighbours. An independent set of G is a set of pairwise non-adjacent vertices and an independent set of largest cardinality is called a maximum independent set. The cardinality of the maximum independent set is called the independence number of G, denoted $\alpha(G)$. This paper concerns a new continuous optimization formulation for the independence number of a graph.

Closely related to the independence number are the concepts of maximality and domination. An independent set is said to be *maximal* if it is not a subset of any larger independent set. Clearly a maximum independent set is maximal but the converse not true in general. A set $S \subseteq V$ is a *dominating set* if every $v \in V \setminus S$ has a neighbour $u \in S$. One can show that every vertex not in a maximal independent set has at least one neighbour in the set, whereby maximal independent sets are also dominating sets.

Computing the independence number of a general graph is NP-complete, although it is known to be solvable in polynomial time for some subclasses, such as claw-free graphs and perfect graphs [20,7]. Computing the independence number is clearly

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a discrete optimization problem. However there are several continuous optimization formulations for this quantity. Perhaps the most well known amongst them is the result by Motzkin and Straus [21] which shows that for a graph G with n vertices,

$$\frac{1}{\alpha(G)} = \min\{x^\top (A+I)x \mid \mathbf{e}^\top x = 1;\ x \geq 0\}, \tag{1}$$

where $\mathbf{e}$ is the vector[1] of 1's in $\mathbb{R}^n$, $A = [a_{ij}]$ is the adjacency matrix of G (i.e., $a_{ij} = 1$ if $(i,j) \in E$ and $= 0$ otherwise), and I is the $n \times n$ identity matrix. Among other continuous formulations, the ones by Harant are noteworthy [10,9]. Specifically, [9, Theorem 7] shows,

$$\alpha(G) = \max\{\mathbf{e}^\top x - \frac{1}{2}x^\top Ax \mid 0 \leq x \leq \mathbf{e}\}.$$

For a given weight vector $w \in \mathbb{R}^n$, the weight of a set $S \subseteq V$ is the quantity $\sum_{i\in S} w_i$. The weighted independence number denoted $\alpha_w(G)$ is the maximum of the weights over all the independent sets, i.e.,

$$\alpha_w(G) := \max\left\{\sum_{i\in S} w_i \mid S \subseteq V \text{ is independent}\right\}.$$

Clearly $\alpha(G) = \alpha_{\mathbf{e}}(G)$.

This paper characterizes the weighted independence number of a graph in terms of the linear complementarity problem (LCP). Given a matrix $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$, LCP(M,q) is the following problem:

$$\text{Find}\quad x = (x_1, x_2 \cdots x_n)^\top \in \mathbb{R}^n \quad \text{such that} \quad \begin{array}{ll} (1) & x \geq 0, \\ (2) & y = Mx + q \geq 0, \\ (3) & y^\top x = 0. \end{array} \tag{LCP(M,q)}$$

Notice that due to the nonnegativity of x and y, the last condition is equivalent to requiring $x_i y_i = 0$ for all i. This requirement is referred to as the *complementarity condition*. A vector x is said to be a solution of LCP(M,q) if it satisfies the above three conditions. LCPs arise naturally in the characterization of equilibria in bimatrix games and several other problems in operations research. We discuss this problem class later in this paper.

For a simple graph[2] $G = (V,E)$ with $|V| = n$ vertices, consider the LCP$(A+I, -\mathbf{e})$, i.e.,

LCP(G)	Find $x \in \mathbb{R}^n$ such that $x \geq 0,\ (A+I)x \geq \mathbf{e},\ x^\top((A+I)x - \mathbf{e}) = 0.$

We refer to this as LCP(G) and its solution set as SOL(G). Let the characteristic vector of a set $S \subseteq V$ be denoted by $\mathbf{1}_S$; it is the vector in $\{0,1\}^n$ whose ith element is 1 *iff* $i \in S$. It is easy to show that if $S^* \subseteq V$ is a maximum independent set in G then $\mathbf{1}_{S^*}$ solves LCP(G). Consequently, we always have,

$$\alpha(G) \leq \max\{\mathbf{e}^\top x \mid x \text{ solves LCP}(G)\}. \tag{2}$$

Our main result in this paper shows that the inequality in (2) is always tight, even for the weighted independence number.

Theorem 1. *For any simple graph G and weight vector $w \geq 0$,*

$$\alpha_w(G) = \max\{w^\top x \mid x \text{ solves LCP}(A+I, -\mathbf{e})\},$$

where A is the adjacency matrix of G, I is the identity matrix and $\mathbf{e}$ is the vector of 1's in $\mathbb{R}^n$.

To note why the above result is not obvious, consider the quantity $\beta(G)$ defined as the size of the smallest maximal independent set in G (also known as minimum independent dominating set; $\beta(G)$ is often referred to as the independent domination number of G). One can show that the characteristic vector of every maximal independent set solves LCP(G) (Lemma 4 in the next section). Hence, analogous to (2), we have

$$\beta(G) \geq \min\{\mathbf{e}^\top x \mid x \text{ solves LCP}(A+I, -\mathbf{e})\}. \tag{3}$$

We show in Section 3.2 that this inequality is in general strict, however, equality is achieved when the graph is a forest (i.e., graph that is a union of disjoint trees). Indeed we have the following theorem.

Theorem 2. *For a forest G,*

$$\beta(G) = \min\{\mathbf{e}^\top x \mid x \text{ solves LCP}(A+I, -\mathbf{e})\},$$

where A is the adjacency matrix of G, I is the identity matrix and $\mathbf{e}$ is the vector of 1's in $\mathbb{R}^n$.

1 Throughout this paper, vectors are column vectors.
2 We consider only simple graphs, i.e., graphs without self loops which means $a_{ii} = 0$ for all $i \in V$.

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