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Note

Weak oddness as an approximation of oddness and resistance in cubic graphs

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ABSTRACT

We consider the question whether the least number ω of odd components in a 2-factor is the same as the least number ω_{w} of odd components in an even factor of a cubic graph. Both invariants appear in the literature under the same name "oddness". We show that these two invariants are different by constructing graphs G satisfying $\rho(G) < \omega_{\mathrm{w}}(G) < \omega(G)$ (where ρ denotes resistance, i.e. the least number of edges that have to be removed from a cubic graph in order to turn it into a 3-edge-colourable graph). In addition, we demonstrate that the difference between any two of those three invariants may be arbitrarily large. Our results imply that if we replace a vertex of a cubic graph with a triangle, then its oddness (ω) may significantly decrease.

1. Oddness and resistance

Cubic graphs naturally fall into two classes depending on whether they do or do not admit a 3-edge-colouring. Besides the trivial family of graphs with bridges, which are trivially uncolourable, there are many examples of 2-edge-connected cubic graphs that do not admit a 3-edge-colouring. Such graphs are called *snarks*; sometimes they are required to satisfy additional conditions, such as cyclic 4-edge-connectivity and girth at least five, to avoid triviality.

For a cubic graph, being 3-edge-colourable is equivalent to having a decomposition into three perfect matchings. There are several important conjectures for which the existence of such a decomposition immediately implies the conjecture being true, in other words, the conjecture can be reduced to snarks. Examples of this are provided by the Tutte's 5-flow conjecture and the cycle double cover conjecture. While both of these problems seem to be exceedingly difficult for snarks in general, they are trivial for 3-edge-colourable cubic graphs. Not only that: they were both proven for snarks that are close to being 3-edge-colourable in a sense, in particular, first for snarks with oddness 2 [2,4] and later at most 4 [3,7].

The part interesting here is that historically, two very similar but still different definitions of oddness appeared. The one used by Huck and Kochol in [2] is based on cycles and even factors. A cycle is a subgraph in which all vertices have even degree. In a cubic graph, it is a collection of disjoint circuits (connected 2-regular subgraphs) and isolated vertices. If a cycle contains all the vertices of the graph, it is called an *even factor*. Huck and Kochol defined oddness as the least possible number of odd components in an even factor.

The other definition, more common at present (e.g. [6,10,7]), is based on 2-factors. Every bridgeless cubic graph has a perfect matching, also called a 1-factor [8]. Its complement is a 2-regular subgraph containing all vertices of the original graph — a 2-*factor*. Oddness is then defined as the least possible number of odd components in a 2-factor.

Albeit being repeatedly discussed among researchers, the question of whether those two definitions are equivalent remained unanswered. The purpose of this article is to settle it by proving that these two notions are different. In order

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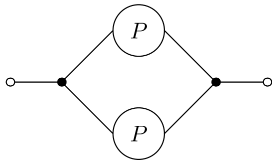

Fig. 1. The 2-pole H (its terminal vertices are marked by empty circles).

to separate them, we will call the one based on even factors *weak oddness*. This choice of terminology is motivated by the fact that an even factor can be viewed as a relaxation of a 2-factor where "degenerate" circuits of length 1 are allowed.

Definition 1. The *oddness* of a bridgeless cubic graph G, denoted by $\omega(G)$, is the least number of odd components in a 2-factor of G.

Definition 2. The *weak oddness* of a cubic graph G, denoted by $\omega_w(G)$, is the least number of odd components in an even factor of G.

Since every 2-factor is an even factor, $\omega_w \leq \omega$.

Another natural measure of uncolourability is based on minimising the use of the fourth colour in a 4-edge-colouring of a cubic graph, which can alternatively be viewed as minimising the number of edges that have to be deleted in order to get a 3-edge-colourable graph. Surprisingly, the required number of edges to be deleted is the same as the number of vertices that have to be deleted in order to get a 3-edge-colourable graph (see [9, Theorem 2.7]). This quantity is called the *resistance* of G, and will be denoted by $\rho(G)$. Observe that $\rho(G) \leq \omega_w(G)$ for every bridgeless cubic graph G since deleting one vertex from each odd circuit in an even factor leaves a colourable graph. It is known that the difference between $\rho(G)$ and $\omega(G)$ can be arbitrarily large in general [10]. We will prove in Section 2 that the same holds for weak oddness.

It is known that if $\rho(G) \leq 2$ (and thus $\omega_w(G) \leq \omega(G) \leq 2$), then $\rho(G) = \omega(G)$ [9, Lemma 2.5] and thus $\omega_w(G) = \omega(G)$. For graphs with resistance more than 2, resistance and oddness may be distinct [10]. In Section 2 we construct a graph with weak oddness 14 and oddness 16 which illustrates that weak oddness and oddness can differ. We do not know, however, whether weak oddness and oddness can be different for graphs with weak oddness smaller than 14. In particular, the following is an open problem.

Question 1. Do there exist cubic graphs with weak oddness 4 and oddness at least 6?

2. Graphs with $\rho < \omega_w < \omega$

Our construction utilises smaller blocks to build larger graphs. A 2-*pole* is a triple (G, s, t), where G is a graph and s, t are two different vertices of G which both have degree one; we will call them *terminals* and the edges incident to them *terminal edges*. Each terminal of a 2-pole serves as a place of connection with another terminal. Two terminal edges of two disjoint 2-poles can be naturally joined to form a new nonterminal edge by identifying the terminal vertices incident to them and suppressing the resulting 2-valent vertex.

A standard way to create terminal vertices is by *splitting off* a vertex v from a graph G; by this we mean removing v from G and attaching a terminal vertex to each dangling edge originally incident with v.

The first step in our construction is to create the 2-pole H depicted in Fig. 1. Let P be the 2-pole obtained from the Petersen graph by inserting a new vertex into one of its edges and then splitting the new vertex off. (The Petersen graph is edge-transitive, so the result of our operation is uniquely determined.) We take two copies (P_1, s_1, t_1), (P_2, s_2, t_2) of P, identify s_1 with s_2, t_1 with t_2, and attach a new terminal edge to both of s_1 and t_1.

The next step is to create the 2-pole H_2 by taking two copies of P and joining a terminal edge of the first copy to a terminal edge of the second copy.

Finally, we take the complete graph on four vertices u_0, u_1, u_2, u_3 and remove all edges incident with u_0. Then, for each $i \in \{1, 2, 3\}$, we take a new copy of H_2 and identify one of its terminals with u_0 and the other with u_i. We denote the resulting cubic graph G (see Fig. 2).

Lemma 3. *If an even factor F of G contains a circuit C passing through a non-terminal vertex of a copy H' of H, but not lying in H', then H' contains at least three odd components of F different from C.*

Proof. If C passes through H, it passes through non-terminal vertices of exactly one copy of P contained in H. In that copy, there is at least one odd component of F apart from C. In the other copy of P, there must be at least two odd components, because the Petersen is not 3-edge-colourable. □