Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Complexity of Grundy coloring and its variants

Édouard Bonnet [a,*,1], Florent Foucaud [b], Eun Jung Kim [c], Florian Sikora [c]

[a] Institute for Computer Science and Control, Hungarian Academy of Sciences (MTA SZTAKI), Hungary
[b] Université Blaise Pascal, CNRS, LIMOS, France
[c] Université Paris-Dauphine, PSL Research University, CNRS, LAMSADE, Paris, France

ARTICLE INFO

Article history:
Received 6 December 2016
Received in revised form 12 December 2017
Accepted 16 December 2017
Available online xxxx

Keywords:
Grundy coloring
Computational complexity
Weak Grundy
Connected Grundy
Exact algorithm
Parameterized complexity

ABSTRACT

The Grundy number of a graph is the maximum number of colors used by the greedy coloring algorithm over all vertex orderings. In this paper, we study the computational complexity of Grundy Coloring, the problem of determining whether a given graph has Grundy number at least k. We also study the variants Weak Grundy Coloring (where the coloring is not necessarily proper) and Connected Grundy Coloring (where at each step of the greedy coloring algorithm, the subgraph induced by the colored vertices must be connected).

We show that Grundy Coloring can be solved in time $O^*(2.443^n)$ and Weak Grundy Coloring in time $O^*(2.716^n)$ on graphs of order n. While Grundy Coloring and Weak Grundy Coloring are known to be solvable in time $O^*(2^{O(wk)})$ for graphs of treewidth w (where k is the number of colors), we prove that under the Exponential Time Hypothesis (ETH), they cannot be solved in time $O^*(2^{o(w \log w)})$. We also describe an $O^*(2^{2^{O(k)}})$ algorithm for Weak Grundy Coloring, which is therefore FPT for the parameter k. Moreover, under the ETH, we prove that such a running time is essentially optimal (this lower bound also holds for Grundy Coloring). Although we do not know whether Grundy Coloring is in FPT, we show that this is the case for graphs belonging to a number of standard graph classes including chordal graphs, claw-free graphs, and graphs excluding a fixed minor. We also describe a quasi-polynomial time algorithm for Grundy Coloring and Weak Grundy Coloring on apex-minor graphs. In stark contrast with the two other problems, we show that Connected Grundy Coloring is NP-complete already for $k = 7$ colors.

1. Introduction

A *k-coloring* of a graph G is a surjective mapping $\varphi : V(G) \to \{1, \dots, k\}$ (we say that vertex v is colored with $\varphi(v)$). A k-coloring φ is *proper* if any two adjacent vertices receive different colors in φ. The *chromatic number* $\chi(G)$ of G is the smallest k such that G has a proper k-coloring. Determining the chromatic number of a graph is one of the most fundamental problems in graph theory. Given a graph G and an ordering $\sigma = v_1, \dots, v_n$ of $V(G)$, the *first-fit coloring algorithm* colors the vertices from v_1 to v_n in the order imposed by σ, and the vertex v_i is colored with the smallest positive integer that is *not* present among the colors of the neighbors of v_i which are in $\{v_1, \dots, v_{i-1}\}$ (in other words, the neighbors of v_i which are already colored). The *Grundy number* $\Gamma(G)$ is the largest k such that G admits a first-fit coloring (for some ordering)

* Corresponding author.
E-mail addresses: edouard.bonnet@ens-lyon.fr (É. Bonnet), florent.foucaud@gmail.com (F. Foucaud), eunjung.kim@dauphine.fr (E.J. Kim), florian.sikora@dauphine.fr (F. Sikora).
[1] Current affiliation: ENS Lyon, LIP, France.

https://doi.org/10.1016/j.dam.2017.12.022

using k colors. First-fit is presumably the simplest heuristic to compute a proper coloring of a graph. In this sense, the Grundy number gives an algorithmic upper bound on the performance of any heuristic for the chromatic number. This notion was first studied by Grundy in 1939 in the context of digraphs and games [4,18], and formally introduced 40 years later by Christen and Selkow [9]. It was independently defined under the name *ochromatic number* by Simmons [36] (the two concepts were proved to be equivalent in [14]). Many works have studied the first-fit algorithm in connection with on-line coloring algorithms, see for example [32]. A natural relaxation of this concept is the *weak Grundy number*, introduced by Kierstead and Saoub [26], where the obtained coloring is not asked to be proper. A more restricted concept is the one of *connected Grundy number*, introduced by Benevides et al. [3], where the algorithm is given an additional "local" restriction on the feasible vertex orderings that can be considered: at each step of the first-fit algorithm, the subgraph induced by the colored vertices must be connected.

The goal of this paper is to advance the study of the computational complexity of determining the Grundy number, the weak Grundy number and the connected Grundy number of a graph.

Let us introduce the problems formally. Let G be a graph and let $\sigma = v_1, \ldots, v_n$ be an ordering of $V(G)$. A k-coloring $\varphi : V(G) \to \{1, \ldots, k\}$ of G is a *first-fit coloring with respect to* σ if for every vertex v_i, the two following conditions hold: (1) for every color (i.e., positive integer) c with $c < \varphi(v_i)$, there is a $j < i$ such that v_i and v_j are adjacent and $\varphi(v_j) = c$, and (2) there is no $j < i$ such that v_i and v_j are adjacent and $\varphi(v_i) = \varphi(v_j)$. A k-coloring is a *Grundy coloring* if it is a first-fit coloring with respect to *some* vertex ordering σ. A k-coloring is a *weak Grundy coloring* if it satisfies the condition (1) with respect to some vertex ordering σ. A vertex ordering $\sigma = v_1, \ldots, v_n$ is *connected* if for every i, $1 \leqslant i \leqslant n$, the subgraph induced by $\{v_1, \ldots, v_i\}$ is connected. A k-coloring is a *connected Grundy coloring* if it is a Grundy coloring with respect to a connected vertex ordering. We note that a (connected) Grundy coloring is a proper coloring, and a weak Grundy coloring is not necessarily proper. Observe that a (connected) Grundy coloring is uniquely defined by its ordering σ, while it is not the case for the weak Grundy coloring.

The maximum number of colors used, taken among all (weak, connected, respectively) Grundy colorings, is called the (*weak, connected*, respectively) Grundy number and is denoted $\Gamma(G)$ ($\Gamma'(G)$ and $\Gamma_c(G)$, respectively). In this paper, we study the complexity of computing these invariants.

Grundy Coloring
Input: A graph G, an integer k.
Question: Do we have $\Gamma(G) \geqslant k$?

Weak Grundy Coloring
Input: A graph G, an integer k.
Question: Do we have $\Gamma'(G) \geqslant k$?

Connected Grundy Coloring
Input: A graph G, an integer k.
Question: Do we have $\Gamma_c(G) \geqslant k$?

Note that $\chi(G) \leqslant \Gamma(G) \leqslant \Delta(G)+1$, where $\chi(G)$ is the chromatic number and $\Delta(G)$ is the maximum degree of G. However, the difference $\Gamma(G) - \chi(G)$ can be (arbitrarily) large, even for bipartite graphs. For example, the Grundy number of the tree of Fig. 1 is 4, whereas its chromatic number is 2. Note that this is not the case for Γ_c for bipartite graphs, since $\Gamma_c(G) \leqslant 2$ for any bipartite graph G [3]. However, the difference $\Gamma_c(G) - \chi(G)$ can be (arbitrarily) large even for planar graphs [3].

Previous results. Grundy Coloring remains NP-complete on bipartite graphs [22] and their complements [38] (and hence claw-free graphs and P_5-free graphs), on chordal graphs [35], and on line graphs [21]. Certain graph classes admit polynomial-time algorithms. There is a linear-time algorithm for Grundy Coloring on trees [23]. This result was extended to graphs of bounded treewidth by Telle and Proskurowski [37], who proposed a dynamic programming algorithm running in time $k^{O(w)}2^{O(wk)}n = O(n^{3w^2})$ for graphs of treewidth w (in other words, their algorithm is in FPT for parameter $k + w$ and in XP for parameter w).[2] A polynomial-time algorithm for Grundy Coloring on P_4-laden graphs, which contains all cographs as a subfamily, was given in [2].

Note that Grundy Coloring admits a polynomial-time algorithm when the number k of colors is fixed [39], in other words, it is in XP for parameter k.

Grundy Coloring has polynomial-time constant-factor approximation algorithms for inputs that are interval graphs [20,32], complements of chordal graphs [20], complements of bipartite graphs [20] and bounded tolerance graphs [26].

[2] The first running time is not explicitly stated in [37] but follows from their meta-theorem. The second one is deduced by the authors of [37] from the first one by upper-bounding k by $w\log_2 n + 1$.

Please cite this article in press as: É. Bonnet, et al., Complexity of Grundy coloring and its variants, Discrete Applied Mathematics (2018), https://doi.org/10.1016/j.dam.2017.12.022.