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# On the chromatic numbers of small-dimensional Euclidean spaces

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ABSTRACT

This paper is devoted to the study of the graph sequence $G_n = (V_n, E_n)$, where $V_n$ is the set of all vectors $v \in \mathbb{R}^n$ with coordinates in $\{-1, 0, 1\}$ such that $|v| = \sqrt{3}$ and $E_n$ consists of all pairs of vertices with scalar product 1. We find the exact value of the independence number of $G_n$. As a corollary we get new lower bounds on $\chi(\mathbb{R}^n)$ and $\chi(\mathbb{Q}^n)$ for small values of $n$.



## 1. Introduction

Let $\mathbb{R}^n$ be the standard Euclidean space, where the distance between any two points $x, y$ is denoted by $|x - y|$. Let $V$ be an arbitrary point set in $\mathbb{R}^n$. Let $a > 0$ be a real number. By a *distance graph* with set of vertices $V$, we mean the graph $G = (V, E)$ whose set of edges $E$ contains *all* pairs of points from $V$ that are at the distance $a$ apart:

$$E = \{\{x, y\} : \ |x - y| = a\}.$$

Distance graphs are among the most studied objects of combinatorial geometry. First of all, they are at the ground of the classical Hadwiger–Nelson problem, which was proposed around 1950 (see [12,27]) and consists in determining the *chromatic number of the space*:

$$\chi(\mathbb{R}^n) = \min\left\{\chi : \ \mathbb{R}^n = V_1 \sqcup \cdots \sqcup V_\chi, \ \forall i \ \forall \mathbf{x}, \mathbf{y} \in V_i \ \ |\mathbf{x} - \mathbf{y}| \neq 1\right\},$$

i.e., the minimum number of colors needed to color all the points in $\mathbb{R}^n$ so that any two points at the distance 1 receive different colors. In other words, it is the chromatic number of the unit distance graph whose vertex set coincides with $\mathbb{R}^n$.

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Due to the extreme popularity of the subject, colorings of unit distance graphs are very deeply explored. Let us just refer the reader to several books and survey articles [21,2,5,14,23,25,24,26,28]. In particular, the best known lower bounds for the chromatic numbers in dimensions $\leqslant 12$ are given below [23,20,8,4,6,18,16,17,15]:

$$\chi(\mathbb{R}^2) \geqslant 4\ [23],\ \chi(\mathbb{R}^3) \geqslant 6\ [20],\ \chi(\mathbb{R}^4) \geqslant 9\ [8],\ \chi(\mathbb{R}^5) \geqslant 9\ [4],\ \chi(\mathbb{R}^6) \geqslant 11\ [6],\ \chi(\mathbb{R}^7) \geqslant 15\ [23],$$

$$\chi(\mathbb{R}^8) \geqslant 16\ [18],\ \chi(\mathbb{R}^9) \geqslant 21\ [16],\ \chi(\mathbb{R}^{10}) \geqslant 23\ [16],\ \chi(\mathbb{R}^{11}) \geqslant 25\ [17],\ \chi(\mathbb{R}^{12}) \geqslant 27\ [15].$$

Recently further improvements were announced [7,13]:

$$\chi(\mathbb{R}^6) \geqslant 12\ [7],\ \chi(\mathbb{R}^7) \geqslant 16\ [7],\ \chi(\mathbb{R}^8) \geqslant 19\ [13],\ \chi(\mathbb{R}^{10}) \geqslant 26\ [7],[13],\ \chi(\mathbb{R}^{11}) \geqslant 32\ [13],\ \chi(\mathbb{R}^{12}) \geqslant 36\ [7].$$

These improvements are essentially based on computer calculations.

In growing dimensions, the following bounds are the best known [22,18]:

$$[22]\ (1.239\ldots + o(1))^n \leqslant \chi(\mathbb{R}^n) \leqslant (3 + o(1))^n\ [18].$$

In this paper, we consider a special sequence of graphs defined in the following way.

Let $V_n$ be the set of all vectors $v$ from $\mathbb{R}^n$ with coordinates in $\{-1, 0, 1\}$ and $|v| = \sqrt{3}$. The set $V_n$ can be considered as the set of vertices of a graph $G_n = (V_n, E_n)$, where an edge connects two vertices if and only if the corresponding vectors have scalar product 1. Note that $G_1$ and $G_2$ are empty and $G_3$ is just a cube.

Recall that an *independent set* in a graph is any set of its vertices which are pairwise non-adjacent and the *independence number* of $G$ denoted by $\alpha(G)$ is the size of a maximum independent set in the graph $G$.

**Theorem 1.** *For $n \geqslant 1$, let $c(n)$ denote the following constant:*

$$c(n) = \begin{cases} 0 & \text{if } n \equiv 0 \\ 1 & \text{if } n \equiv 1 \\ 2 & \text{if } n \equiv 2 \text{ or } 3 \end{cases} \pmod 4.$$

*Then, the independence number of $G_n$ is given by the formula*

$$\alpha(G_n) = \max\{6n - 28, 4n - 4c(n)\}.$$

Actually, the result of Theorem 1 is a far-reaching generalization of a much simpler lemma proved by Zs. Nagy (see [19]) in 1972 and used not only in combinatorial geometry, but also in Ramsey theory. In this lemma, $G'_n = (V'_n, E'_n)$, where $V'_n$ is the set of all vectors $v$, $|v| = \sqrt{3}$, with coordinates in $\{0, 1\}$ and again an edge connects two vertices if and only if the corresponding vectors have scalar product 1. Lemma states that in this case $\alpha(G'_n) = n - c(n)$.

Larman and Rogers used the mentioned lemma to prove $\chi(\mathbb{R}^n) \geqslant (1 + o(1))n^2/6$ (in fact, it was suggested by Erdős and Sós), which was the first nontrivial lower bound on $\chi(\mathbb{R}^n)$. It is worth noting that the chromatic number of $G'_n$ almost coincides with the bound $n/\alpha(G'_n)$, as was shown in [1].

On the other hand there is a natural bijection between $\{0, 1\}^n$ and the subsets of $n$-element set, which gives deep combinatorial sense to graphs of the mentioned types. In several recent papers [9,11,10] Frankl and Kupavskii consider analogues of some classical combinatorial problems in $\{0, \pm 1\}$ setup.

The proof of Theorem 1 is given in the following parts: some examples showing the lower bound in Theorem 1 and some preliminaries are given in Section 2; the upper bound is proved in Section 3 (for the case $n \leqslant 13$ we use computer simulations). Note that, roughly speaking, the quantity 13 is a threshold where the bound $6n - 28$ starts dominating the bound $4n$.

As a corollary of Theorem 1 we get the following bounds for the chromatic numbers of Euclidean spaces.

**Theorem 2.** *Let $c(n)$ be the constant defined in Theorem 1. Then, for all $n \geqslant 3$, we have*

$$\chi(\mathbb{R}^n) \geqslant \chi(\mathbb{Q}^n) \geqslant \chi(G_n) \geqslant \frac{|V_n|}{\alpha(G_n)} = \frac{8\binom{n}{3}}{\max\{6n - 28, 4n - c(n)\}}.$$

Asymptotically, the bound in this theorem is $\frac{2}{9}n^2(1 + o(1))$, which is a weak result. On the other hand, for small values of $n$, the theorem gives the best known bounds, namely:

$$\chi(\mathbb{R}^9) \geqslant \chi(\mathbb{Q}^9) \geqslant 21,$$

$$\chi(\mathbb{R}^{10}) \geqslant \chi(\mathbb{Q}^{10}) \geqslant 30,$$

$$\chi(\mathbb{R}^{11}) \geqslant \chi(\mathbb{Q}^{11}) \geqslant 35,$$

$$\chi(\mathbb{R}^{12}) \geqslant \chi(\mathbb{Q}^{12}) \geqslant 37.$$

Actually, we will show in Section 4 the following stronger result for $n = 9$.