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On distance-preserving elimination orderings in graphs: Complexity and algorithms[☆]

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A B S T R A C T

For every connected graph G, a subgraph H of G is *isometric* if the distance between any two vertices in H is the same in H as in G. A *distance-preserving elimination ordering* of G is a total ordering of its vertex-set $V(G)$, denoted $(v_1, v_2, \ldots, v_n)$, such that any subgraph $G_i = G \setminus (v_1, v_2, \ldots, v_i)$ with $1 \leq i < n$ is isometric. This kind of ordering has been introduced by Chepoi in his study on weakly modular graphs (Chepoi, 1998). We prove that it is NP-complete to decide whether such ordering exists for a given graph — even if it has diameter at most 2. Then, we prove on the positive side that the problem of computing a distance-preserving ordering when there exists one is fixed-parameter-tractable in the treewidth. Lastly, we describe a heuristic in order to compute a distance-preserving ordering when there exists one that we compare to an exact exponential time algorithm and to an ILP formulation for the problem.

1. Introduction

Elimination orderings of a graph are total orderings of its vertex-set. Many interesting graph problems can be specified in terms of the existence of an elimination ordering with some given properties. These range from some practical problems in molecular biology and chemistry [8] to the analysis of graph search algorithms [14], the characterization of some graph classes [10,29], and the study of network clustering methods in social networks [26]. On the computational point of view, vertex ordering characterizations of a given graph class often lead to efficient (polynomial-time) recognition algorithms for the graphs in this class [2,6,15,21,28]. In this work we will consider one specific kind of elimination ordering that is called *distance-preserving elimination ordering* [11]. Precisely, let us remind that a subgraph H of a graph G is *isometric* if the distance between any two vertices in H is the same in H as in G. An elimination ordering $(v_1, v_2, \ldots, v_n)$ of G is distance-preserving if it satisfies that each suffix $(v_i, v_{i+1}, \ldots, v_n)$ with $i < n$ induces an isometric subgraph of G.

Distance-preserving elimination orderings encompass several other elimination orderings studied in the literature [6,7,19,24,25,28], all of which can be computed in polynomial time when they exist. In particular, known refinements of distance-preserving elimination orderings comprise the perfect elimination orderings [28], maximum neighbourhood orderings [6], h-extremal orderings [7], semisimplicial elimination orderings [24], dismantlable orderings [25] and more generally *domination elimination orderings* [19]. The latter orderings characterize chordal graphs, dually chordal graphs,

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homogeneously orderable graphs, cop-win graphs and a subclass of tandem-win graphs [12], respectively, and as above stated they all can be computed in polynomial-time when they exist. However the complexity of deciding whether a distance-preserving elimination ordering exists in a given graph has been left open until this paper. We aim at completing the picture and characterizing the complexity of this problem.

Related work. In [17] it has been proved that every graph with a distance-preserving elimination ordering has a *minimum-size cycle basis* with only triangles and quadrangles, that can be easily computed if a distance-preserving elimination ordering is part of the input. This property has been useful in the study of some tree-likeness invariants of graphs (*e.g.*, in comparing treewidth with treelength). However, the complexity of recognizing graphs with a distance-preserving elimination ordering has been left open in [17]. Prior works [9,11] have focused on the existence of distance-preserving elimination orderings in some well-structured graph classes, *i.e.*, the *weakly modular graphs*. In particular, it has been proved recently in [9] that every breadth-first search ordering of a weakly modular graph is distance-preserving, that allows to compute one such ordering in linear time for a given graph in this class.

On the positive side, above stated refinements of distance-preserving elimination orderings [6,7,19,24,25,28] can all be computed with greedy algorithms when they exist. Indeed, for all these orderings it can be tested in polynomial-time whether a given vertex can be eliminated first. As an example, any dominated vertex can be the starting vertex of some domination elimination ordering (total ordering of the vertex-set where for every suffix, the closed neighbourhood of the first vertex is dominated in the subgraph induced by the suffix). The latter implies that any partial domination elimination ordering can be extended unless the graph does not admit such a total order. A first hint that computing a distance-preserving elimination ordering can be more difficult is that it is not that simple to choose a starting vertex. For instance, consider the wheel W_5 obtained from a cycle C_5 of length five by adding a universal vertex. Every elimination ordering of W_5 where the universal vertex is the last vertex eliminated is distance-preserving. However, if the universal vertex is eliminated first then the cycle C_5 is an isometric subgraph of W_5 that does not admit a distance-preserving elimination ordering. The above problem occurring with C_5 also occurs with *hypercubes*, that can be proved using tools from discrete geometry.[1]

Our contributions. We prove on the negative side that it is NP-complete to decide whether a given graph admits a distance-preserving elimination ordering (Section 3). The latter result may look surprising since as above stated, a broad range of distance-preserving orderings with additional properties can be computed in polynomial time when they exist. Then we show that the problem remains NP-complete even for general graphs with diameter at most two (Section 3.3). Note that in a sense our result is optimal w.r.t. the diameter because complete graphs trivially admit a distance-preserving ordering. Our reduction will show how to encode a 3-SAT formula in a graph whose distance-preserving orderings are in many-to-many correspondence with the satisfying assignments for the formula. This line of work resembles to the one in [31] in order to show that it is NP-complete to recognize collapsible complexes. Our work differs from theirs in that we study orderings with very distinct properties and the "simpler" structure of graphs – w.r.t. complexes – further constrains our gadgets to mimic variables and clauses of the formula.

On a more positive side, we prove in Section 4 that the problem of computing a distance-preserving ordering when there exists one is fixed-parameter-tractable in the treewidth.

Next, we show that a meta-theorem on vertex-orderings [3] can be applied to our problem, that results in an algorithm with $\mathcal{O}^*(2^n)$-time and space complexity, as well as in an algorithm with $\mathcal{O}^*(4^n)$-time and polynomial space complexity. We also propose an Integer Linear Programming formulation which may lead to a better running time in practice. These exact algorithms are described in Section 5 as well as simple heuristic algorithms.

Notations. Graphs in this study are finite, simple (hence without loop nor multiple edges) and unweighted. We refer to [5,20] for standard reference books on graphs (see also [1] for a survey about metric graph theory). Let $(v_1, v_2, \ldots, v_n)$ be an elimination ordering of a graph G, we say that vertex v_i, $1 \leq i \leq n$, is the ith vertex to be eliminated, and that vertex v_i is eliminated before vertex v_j, denoted $v_i \prec v_j$, if $i < j$.

2. Local characterization

In what follows, we will avoid considering all the distances in the graph at each time a vertex is eliminated. That is, we replace the "global" condition that $G \setminus v$ is isometric by a "local" one implying only the neighbours of v. The following characterization will explain how to do so.

Lemma 1. *Let $G = (V, E)$ and $v \in V$, the subgraph $G \setminus v$ is isometric if and only if every two non-adjacent neighbours of vertex v have at least two common neighbours in G (including v).*

Proof. If $G \setminus v$ is isometric, then let $x, y \in N_G(v)$ be non-adjacent. Since $d_{G\setminus v}(x, y) = d_G(x, y) = 2$, x and y have another common neighbour than vertex v. Conversely, suppose that every two non-adjacent neighbours of vertex v have at least

[1] More precisely, for the special case of an n-dimensional hypercube, the distance-preserving orderings are equivalent to the so-called "shellable orderings" as defined in [32]. In particular, if every partial distance-preserving ordering of the n-dimensional hypercube could be extended, then it would imply that its dual, the n-dimensional octahedron, is extendably shellable, that is known to be false for $n \geq 12$ [22].