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On a four-parameter generalization of some special sequences

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ABSTRACT

We introduce a new four-parameter sequence that simultaneously generalizes some well-known integer sequences, including Fibonacci, Padovan, Jacobsthal, Pell, and Lucas numbers. Combinatorial interpretations are discussed and many identities for this general sequence are derived. As a consequence, a number of identities for Fibonacci, Lucas, Pell, Jacobsthal, Padovan, and Narayana numbers as well as some of their generalizations are obtained. We also present the Cassini formula for the new sequence.

1. Introduction

The Fibonacci numbers are defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with initial conditions $F_0 = 0$ and $F_1 = 1$. The Fibonacci numbers have been generalized by a number of authors in many different ways. In [13], the authors introduced the k-Fibonacci numbers $F(k, n) = kF(k, n-1) + F(k, n-2)$, for $n \geq 2$, $F(k, 0) = 0$ and $F(k, 1) = 1$. In [27], the $(2, k)$-Fibonacci numbers were introduced by the recurrence relation $F_2(k, n) = F_2(k, n-2) + F_2(k, n-k)$, for $n \geq k$, with $F_2(k, n) = 1$ for $n = 0, \ldots, k-1$, and the authors interpreted them in terms of certain set decompositions. A two-variables polynomial generalization is presented in [1]. Among others, some generalizations can be seen in [2,3,5,12,14,18]. Some of the generalizations have interesting and useful graph interpretations, see [24–26].

In this paper, we introduce a new generalization not only for the Fibonacci numbers, but for many other special integer sequences and their generalizations. Our four-parameter sequence, $F^i_{r,s}(k, n)$, is defined by the recurrence relation:

$$F^i_{r,s}(k, n) = rF^i_{r,s}(k, n-i) + sF^i_{r,s}(k, n-k),$$

with appropriate initial conditions. We also introduce a new generalization of the Lucas numbers L_n:

$$L^i_{r,s}(k, n) = r^{k-1}\{(k-1)sF^i_{r,s}(k, n-k-(k-1)i) + F^i_{r,s}(k, n-(k-1)i)\}.$$

For $i = r = s = 1$ and $k = 2$ in $F^i_{r,s}(k, n)$ and $L^i_{r,s}(k, n)$, we obtain the numbers F_n and L_n, respectively. In addition, Pell, Jacobsthal, Padovan and Narayana numbers and some of their generalizations are obtained from $F^i_{r,s}(k, n)$ (Section 2).

We present three combinatorial interpretations for $F^i_{r,s}(k, n)$: one in terms of tilings, another in terms of color decomposition of sets and, finally, a graph interpretation. A tiling interpretation is also given for $L^i_{r,s}(k, n)$.

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We use the tiling counting technique, to obtain a closed formula for $F^i_{r,s}(k,n)$ and new identities for these sequences. The combinatorial aspects of counting via tilings has been explored and employed by many authors, see [5,6,8,11] for instance. Although we could have just used induction in the proofs, we preferred the tilling approach in order to make it clear where the identities stem from.

In Section 5 we are describe the matrix from where we can obtain the numbers $F^i_{r,s}(k,n)$. In other words, we develop to $F^i_{r,s}(k,n)$ the matrix approach that has been used by many authors for studying particular sequences, see for instance [9,27].

2. The sequences $F^i_{r,s}(k,n)$ and $L^i_{r,s}(k,n)$

In this section, we introduce the numbers $F^i_{r,s}(k,n)$ and $L^i_{r,s}(k,n)$ and provide examples of sequences that can be obtained from them by specializing the parameters r, s, i, and k. In particular, these new numbers are generalizations of the Fibonacci and Lucas numbers.

Let $i, k, r, s \geq 1$, and $n \geq -1$ be integers. We define the sequences $F^i_{r,s}(k,n)$ and $L^i_{r,s}(k,n)$ by the following recurrence relations:

$$F^i_{r,s}(k,n) = \begin{cases} rF^i_{r,s}(k,n-i) + sF^i_{r,s}(k,n-k), & \text{for } n \geq \max\{k,i\}-1 \\ r^{\lfloor \frac{n+1}{i} \rfloor}, & \text{for } i \leq k \text{ and } n = -1,0,1,\ldots,k-2 \\ s^{\lfloor \frac{n+1}{k} \rfloor}, & \text{for } k < i \text{ and } n = -1,0,1,\ldots,i-2 \end{cases} \tag{1}$$

and

$$L^i_{r,s}(k,n) = r^{k-1}\{(k-1)sF^i_{r,s}(k,n-k-(k-1)i) + F^i_{r,s}(k,n-(k-1)i)\},$$

for $k > i$ and $n \geq k+(k-1)i-1$, with initial conditions:

$$L^i_{r,s}(k,n) = \begin{cases} F^i_{r,s}(k,n), & \text{for } n = -1,0,\ldots,k-2 \\ F^i_{r,s}(k,n) - r^{\lfloor \frac{n+1}{i} \rfloor}, & \text{for } n = k-1,\ldots,k+(k-1)i-2. \end{cases}$$

The Fibonacci sequence can be obtained directly from $F^i_{r,s}(k,n)$ in two different ways: either by taking $r=s=i=1$ and $k=2$ by taking $r=s=k=1$ and $i=2$, i.e., $F^1_{1,1}(2,n) = F^2_{1,1}(1,n) = F_{n+2}$ for $n \geq -1$. We also note that $L^1_{1,1}(2,n) = L_n$, for $n \geq 2$, the nth Lucas number. Table 1 exhibits some other well-known sequences that can be derived from $F^i_{r,s}(k,n)$ and $L^i_{r,s}(k,n)$.

Remark 2.1. As we are considering that i, k, r, and s are nonnegative integers, the polynomial generalization of the Fibonacci numbers $\{n\}_{s,t}$ presented in [1] and the two-parameters sequence $p^{a,b}_n$ from [5] coincide as we set $a=s$ and $b=t$. In Section 4 new identities for $\{n\}_{s,t}$, and so for $p^{a,b}_n$, are obtained.

Remark 2.2. In order to avoid confusion, the generalizations of the Fibonacci numbers presented in [20,27], and [3] are denoted here by $F_1(k,n)$, $F_s(k,n)$, and $F_{k-1}(k,n)$, respectively.

We finish this section by presenting a recurrence relation for $L^i_{r,s}(k,n)$.

Theorem 2.1. *Let $k > i$ and $n \geq 2k+(k-1)i-1$ be integers. Then,*

$$L^i_{r,s}(k,n) = rL^i_{r,s}(k,n-i) + sL^i_{r,s}(k,n-k).$$

Proof. Using (1) and the definition of $L^i_{r,s}(k,n)$, we have

$$\begin{aligned} L^i_{r,s}(k,n) &= (k-1)r^{k-1}sF^i_{r,s}(k,n-k-(k-1)i) + r^{k-1}F^i_{r,s}(k,n-(k-1)i) \\ &= (k-1)r^k sF^i_{r,s}(k,n-k-(k-1)i-i) \\ &\quad +(k-1)r^{k-1}s^2F^i_{r,s}(k,n-k-(k-1)i-k) \\ &\quad +r^kF^i_{r,s}(k,n-(k-1)i-i) + r^{k-1}sF^i_{r,s}(k,n-(k-1)i-k), \end{aligned}$$

from where

$$\begin{aligned} L^i_{r,s}(k,n) &= (k-1)r^k sF^i_{r,s}(k,n-k-ki) + (k-1)r^{k-1}s^2F^i_{r,s}(k,n-2k-ki+i) \\ &\quad +r^kF^i_{r,s}(k,n-ki) + r^{k-1}sF^i_{r,s}(k,n-ki+i-k). \end{aligned}$$