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Distribution centers in graphs

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ABSTRACT

For a graph $G = (V, E)$ and a set $S \subseteq V$, the boundary of S is the set of vertices in $V \setminus S$ that have a neighbor in S. A non-empty set $S \subseteq V$ is a distribution center if for every vertex v in the boundary of S, v is adjacent to a vertex in S, say u, where u has at least as many neighbors in S as v has in $V \setminus S$. The distribution center number of a graph G is the minimum cardinality of a distribution center of G. We introduce distribution centers as graph models for supply–demand type distribution. We determine the distribution center number for selected families of graphs and give bounds on the distribution center number for general graphs. Although not necessarily true for general graphs, we show that for trees the domination number and the maximum degree are upper bounds on the distribution center number.

1. Introduction

A distribution center for a set of products is a structure or a group of units used to store goods that are to be distributed to retailers, to wholesalers, or directly to consumers. Distribution centers are usually thought of as being demand driven. In this paper we consider a graph model for a distribution center, specifically, a set of vertices which models a supply and demand situation.

We begin with some basic terminology. Let $G = (V, E)$ be a graph having *order* $n = |V|$ vertices. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v is $|N(v)|$ and is denoted by $\deg(v)$. A vertex with degree one is called a *leaf*, and its only neighbor is called a *support vertex*. The *open neighborhood of a set* $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood of a set* S is the set $N[S] = N(S) \cup S = \cup_{v\in S} N[v]$. The *boundary* of S, denoted $\partial(S)$ is $\partial(S) = N(S) \cap (V \setminus S)$, that is, the boundary is the set of vertices in $V \setminus S$ that are adjacent to at least one vertex in S. The S-*private neighborhood* of a vertex $v \in S$ is the set $N[v] \setminus (N[S \setminus \{v\}])$; vertices in this set are called *private neighbors* of v (with respect to S). Let $G[S]$ denote the subgraph induced in G by the set $S \subseteq V$. A set $S \subseteq V$ in a graph G is called a *dominating set* if $N[S] = V$ and a *total dominating set* if $N(S) = V$. The *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$) equals the minimum cardinality of a dominating set (respectively, total dominating set) in G. A dominating set of G with cardinality $\gamma(G)$ is a called γ-set of G, and similar notation is used for other parameters. The *corona* $G \circ K_1$ of G is the graph formed from G by adding for each $v \in V$, a new vertex v' and edge vv'.

Let $[A, B]$ denote the set of all edges between a vertex in A and a vertex in B.

We are now ready to define distribution centers in graphs.

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Definition 1. A non-empty set of vertices $S \subseteq V$ is a *distribution center* if every vertex $v \in \partial(S)$ is adjacent to a vertex $u \in S$ with $|N[u] \cap S| \geq |N[v] \cap (V \setminus S)|$. The minimum cardinality of a distribution center of a graph G is the *distribution center number* $\text{dc}(G)$, and a distribution center of G with cardinality $\text{dc}(G)$ is called a dc-set of G.

Let S be a non-empty set of vertices of G. If $u \in S$, $v \in V \setminus S$ and $v \in N(u)$, such that $|N[u] \cap S| \geq |N[v] \cap (V \setminus S)|$, then we say that u supplies the demand of v, or equivalently, v is supplied by u. One way to look at a distribution center S is to think of a vertex $v \in \partial(S)$ and its neighbors in $V \setminus S$ as needing some amount of resource units, one unit per vertex, while each vertex in S is able to supply one unit of the resource. Thus, a vertex in $\partial(S)$ makes a demand on the distribution center S and is supplied by one of its neighbors in S. Vertex v can ask a vertex $u \in S \cap N(v)$ to deliver $|N[v] \cap (V \setminus S)|$ units. Vertex u can provide this amount only if vertex u can receive from itself and its neighbors in S at least this number, i.e., $|N[u] \cap S| \geq |N[v] \cap (V \setminus S)|$. We can think of such a set S as a distribution center that is capable of providing *two-day delivery* to any vertex (customer) in $\partial(S)$: on day-one, each neighbor of $u \in S$ can ship one unit of resource to u, and then, on day-two, vertex u can ship $|N[v] \cap V \setminus S|$ units of resource to its neighbor $v \in \partial(u)$.

A related concept of an offensive alliance was defined in [2] as follows.

Definition 2. A non-empty set of vertices $S \subseteq V$ is an *offensive alliance* if for every vertex $v \in \partial(S)$, $|N(v) \cap S| \geq |N[v] \cap (V \setminus S)|$.

Many applications of alliances, including the coalition of nations for defense purposes, were stated in [2]. From the perspective of a supply–demand distribution center application, we note that an offensive alliance S can provide vertex $v \in \partial(S)$ *one day delivery*, that is, S can provide vertex v the resources it wants distributed over the edges between vertices in S and v. Although distribution centers and alliances are similar concepts, the corresponding parameters can easily be shown to be incomparable.

To illustrate distribution centers, we consider a few examples. A *star* on n vertices is the complete bipartite graph $K_{1,n-1}$, and the *double star* $S_{p,q}$ is the tree with exactly two adjacent non-leaf vertices, one of which is adjacent to $p \geq 1$ leaves and the other to $q \geq 1$ leaves. Note that $\text{dc}(K_{1,n-1}) = 1$ since each leaf has a demand of one and the center vertex can supply the demand. Similarly, the set of two center vertices is a distribution center and no single vertex is a distribution center for $S_{p,q}$, so, $\text{dc}(S_{p,q}) = 2$. For another example, consider the path $P_{11} = (v_1, v_2, \ldots, v_{11})$. The set $S = \{v_3, v_4, v_8, v_9\}$ is a distribution center. Note that the boundary of S is the set $\{v_2, v_5, v_7, v_{10}\}$, and the boundary of S does not include vertices v_1, v_6, or v_{11}. This example shows the following:

(i) A distribution center need not be a dominating set, and the vertices not in the boundary of a distribution center do not have to have their demands met.
(ii) A distribution center need not induce a connected subgraph.
(iii) A subset of a distribution center can also be a distribution center, since both $\{v_3, v_4\}$ and $\{v_8, v_9\}$ are distribution centers, that is, a distribution center need not be a minimal distribution center.
(iv) A graph can have several vertex-disjoint distribution centers.

If a distribution center is also a dominating set, i.e., $\partial(S) = V \setminus S$, then S is called a *global distribution center*. Formally, we give the following definition.

Definition 3. A set $S \subseteq V$ is a *global distribution center* if every vertex $v \in V \setminus S$ is adjacent to a vertex $u \in S$ with $|N[u] \cap S| \geq |N[v] \cap (V \setminus S)|$. The *global distribution center number* $\text{gdc}(G)$ is the minimum cardinality of a global distribution center of G. A global distribution center with cardinality $\text{gdc}(G)$ is called a gdc-set of G.

Clearly, every global distribution center is a dominating set. Moreover, every graph G has a distribution center and a global distribution center since the set $V(G)$ is trivially both of these. Hence, $\text{dc}(G)$ and $\text{gdc}(G)$ are defined and we have the following observation.

Observation 4. *For any graph G, $\gamma(G) \leq \text{gdc}(G)$ and $\text{dc}(G) \leq \text{gdc}(G) \leq |V(G)|$.*

As we will see, all of these bounds can be both strict and tight.

Our main focus in this paper is to introduce and investigate distribution centers and global distribution centers. Since the distribution center number of a disconnected graph is the minimum distribution center number of any of its connected components, and the global distribution center number is the sum of the global distribution center numbers of its components, we consider only connected graphs. In Section 2, we begin by characterizing graphs having small distribution center numbers and determining values of the distribution center number of specific families of graphs. Bounds on the distribution center numbers are presented in Section 3. Among other bounds, we show that cubic graphs have distribution center numbers which are bounded above by their girth. We also show that the distribution center number of a non-trivial tree is bounded above by its domination number and by its maximum degree. We conclude with a list of open problems in Section 4.

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