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Efficient eight-regular circulants based on the Kronecker product

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ABSTRACT

This paper presents a family of efficient eight-regular circulants representable as the Kronecker product of the dense four-regular circulant on $2r^2 + 2r + 1$ nodes and the cycle C_{4r+3}, where $r \equiv 0, 1, 2, 4 \pmod 5$. Each graph is of order $\frac{1}{2}(2d+1)(d^2+1)$, where d denotes its diameter, d is odd, and $d \equiv 0, 1, 3, 4 \pmod 5$. Its average distance is about two-thirds of its diameter. Other salient characteristics include high odd girth, three-colorability, and an edge decomposition into Hamiltonian cycles. The baseline of the present study is a theorem by Broere and Hattingh, which states that the Kronecker product of two circulants whose orders are co-prime is a circulant itself.

1. Introduction

Circulant graphs possess a number of attractive features such as high symmetry, high connectivity and scalability, based on which they are amenable to an application as a *network topology* in areas like parallel computers, distributed systems and VLSI [1,9].

Over the years, the four-regular circulants have received a lot of attention. See the surveys [1,9,17] and the references therein. On the other hand, higher-degree circulants have not been studied at length. This is probably because problems in the latter case are relatively challenging. For example, the diameter of a four-regular circulant on n vertices is greater than or equal to $\left\lceil \frac{1}{2}\left(-1+\sqrt{2n-1}\right)\right\rceil$, a bound achievable by graphs in a number of families [18]. By contrast, the situation is not so tractable in the case of higher-degree circulants [8].

The author [11] recently presented a family of six-regular circulants representable as the Kronecker product of the Möbius ladder on p vertices and the cycle C_{p+3}, where $p \equiv 4, 8 \pmod{12}$. It turns out that these graphs outperform the well-known triple-loop networks [19].

This paper presents a family of *eight-regular circulants* representable as the *Kronecker product* of the *dense four-regular circulant* on $2r^2 + 2r + 1$ vertices and the *cycle* C_{4r+3}, where $r \equiv 0, 1, 2, 4 \pmod 5$. Each graph is such that (i) its odd girth is equal to $2d+1$, d being the diameter, (ii) its average distance is about two-thirds of the diameter, (iii) its chromatic number is equal to three, and (iv) it admits a Hamiltonian decomposition. (High odd girth, low average distance, low chromatic number and Hamiltonian decomposition are a big plus in a network.) For certain previous studies on eight-regular circulants, see Dougherty and Faber [5], Lewis [15], and Kreher and Westlund [14].

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1.1. Definitions and preliminaries

When we speak of a graph, we mean a finite, simple, undirected and connected graph. Let G be a graph, and let $d_G(u, v)$ denote the shortest distance between vertices u and v in G. Further, let dia(G) represent its *diameter*, i.e., $\max\{d_G(u, v): u, v \in V(G)\}$. A distance-preserving subgraph of a graph is called an *isometric subgraph* [6]. We employ *vertex* and *node* as synonyms, and write $G \cong H$ if G is isomorphic to H.

Say that a vertex v in G is at *level i* relative to a fixed vertex u if $d_G(u, v) = i$. A *level diagram* of G relative to u consists of a layout of the graph in which vertices at a distance of i from u appear on a "line at a height" of i above u, for $0 \leq i \leq \text{dia}(G)$. Vertices at a distance of dia(G) from u are called *diametrical* relative to u. If G is known to be vertex-transitive, a property held by a circulant [7], then the form of its level diagram is independent of the choice of the source vertex.

The *Kronecker product* (also known as the tensor product, direct product, etc.) $G \times H$ of graphs $G = (U, D)$ and $H = (W, F)$ is defined as follows: $V(G \times H) = U \times W$, and $E(G \times H) = \{\{(a, x), (b, y)\} \mid \{a, b\} \in D \text{ and } \{x, y\} \in F\}$. It is one of the most important products, with numerous applications in areas such as computer networks, perfect codes and algebraic systems [6].

Let C_n denote the cycle on the vertex set $\{0, \ldots, n-1\}$, $n \geq 3$, where adjacencies $\{i, i+1\}$ exist in the natural way. A spanning cycle in a graph (if one exists) is called a *Hamiltonian cycle*. Further, a graph is said to admit a *Hamiltonian decomposition* if its edge set may be partitioned into Hamiltonian cycles. The length of a shortest (induced) odd cycle in a nonbipartite graph G is called its *odd girth*, denoted by $og(G)$. Let $\chi(G)$ denote the *chromatic number* of G. For undefined terms, see Hammack et al. [6].

Proposition 1.1 ([6,4]).

(1) *$G \times H$ is connected iff both G and H are connected and at least one of them is nonbipartite.*
(2) *The degree of a vertex (u, v) in $G \times H$ is equal to the product of the degrees of u and v in G and H, respectively.*
(3) $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.
(4) *If G and H are both nonbipartite, then $og(G \times H) = \max\{og(G), og(H)\}$.*
(5) *If G and H are both vertex-transitive, then so is $G \times H$.* □

Let $n, s_1, \ldots, s_k$ be such that $n \geq 3$, and $1 \leq s_1 < s_2 < \cdots < s_k \leq \lfloor n/2 \rfloor$. The *circulant* $\mathscr{C}_n(s_1, \ldots, s_k)$ is a graph on the vertex set $\{0, \ldots, n-1\}$, where each vertex i is adjacent to each of $(i \pm s_1) \bmod n, \ldots, (i \pm s_k) \bmod n$. The parameters $s_1, \ldots, s_k$ are called the *step sizes* or *jumps*. If n is even and $s_k = n/2$, then the graph is $(2k-1)$-regular, otherwise it is $2k$-regular. It is known that [3] (i) $\mathscr{C}_n(s_1, \ldots, s_k)$ is connected iff $\gcd(n, s_1, \ldots, s_k) = 1$, and (ii) $\mathscr{C}_n(s_1, \ldots, s_k)$ is bipartite iff n is even, and each of $s_1, \ldots, s_k$ is odd.

Here is the baseline of the present study.

Proposition 1.2 ([4]). *If G and H are circulants whose orders are co-prime, then $G \times H$ is a circulant itself.* □

1.2. Distances in the Kronecker product

The distance in the Kronecker product is governed by a formula that is based on a shortest even walk and a shortest odd walk (and the respective even distance and the odd distance) between two vertices in the factor graphs [13]. To that end, let $d^e_G(a, b)$ and $d^o_G(a, b)$ denote the shortest even distance and the shortest odd distance, respectively, between vertices a and b in a graph G. (If G is nonbipartite, then the two parameters are finite in respect of every pair of vertices.)

Proposition 1.3 ([13]). *If G and H are both nonbipartite graphs, then $d_{G\times H}((a, x), (b, y)) = \min\{\max\{d^e_G(a, b), d^e_H(x, y)\}, \max\{d^o_G(a, b), d^o_H(x, y)\}\}$.* □

Corollary 1.4 ([13]). *If a and b are both odd and $a \geq b$, then*

$$dia(C_a \times C_b) = \begin{cases} a-1 & \text{if } a = b \\ \max\{\frac{1}{2}(a-1), b\} & \text{if } a > b. \end{cases} \quad \square$$

What follows

Section 2 introduces the dense four-regular circulant $\mathscr{C}_{2r^2+2r+1}(1, 2r+1)$, while Section 3 presents the actual family of eight-regular circulants. The distance-wise node distributions of the resulting graphs appear next, while their jump sequences appear in Section 5.

See Table 1 for the essential nomenclature.