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Rainbow connections in digraphs

Elżbieta Sidorowicz [a,*], Éric Sopena [b]

[a] Faculty of Mathematics, Computer Science and Econometrics, University of Zielona Góra, Zielona Góra, Poland
[b] Univ. Bordeaux, Bordeaux INP, CNRS, LaBRI, UMR5800, F-33400 Talence, France

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ABSTRACT

An arc-coloured path in a digraph is rainbow if its arcs have distinct colours. A vertex-coloured path is vertex rainbow if its internal vertices have distinct colours. A totally-coloured path is total rainbow if its arcs and internal vertices have distinct colours. An arc-coloured (resp. vertex-coloured, totally-coloured) digraph D is rainbow connected (resp. rainbow vertex-connected, total rainbow connected) if any two vertices of D are connected by a rainbow (resp. vertex rainbow, total rainbow) path. The rainbow connection number (resp. rainbow vertex-connection number, total rainbow connection number) of a digraph D is the smallest number of colours needed to make D rainbow connected (resp. rainbow vertex-connected, total rainbow connected).

In this paper, we study the rainbow connection, rainbow vertex-connection and total rainbow connection numbers of digraphs. We give some properties of these parameters and establish relations between them. The rainbow connection number and the rainbow vertex-connection number of a digraph D are both upper bounded by the order of D, while its total rainbow connection number is upper bounded by twice of its order. In particular, we prove that a digraph of order n has rainbow connection number n if and only if it is Hamiltonian and has three vertices with eccentricity $n-1$, that it has rainbow vertex-connection number n if and only if it has a Hamiltonian cycle C and three vertices with eccentricity $n-1$ such that no two of them are consecutive on C, and that it has total rainbow connection number $2n$ if and only if it has rainbow vertex-connection number n.

1. Introduction

We consider finite and simple graphs only, and refer to [5] for terminology and notation not defined here.

In an edge-coloured graph G, a path is said to be *rainbow* if it does not use two edges with the same colour. Then the graph G is said to be *rainbow connected* if any two vertices are connected by a rainbow path. The *rainbow connection number* of G, denoted by $\mathrm{rc}(G)$, is the smallest possible number of colours in a rainbow connected colouring of G.

The concept of rainbow connection in graphs was introduced by Chartrand, Johns, McKeon and Zhang in [8]. Since then, the rainbow connection number has attracted much attention. The rainbow connection number of some special graph classes was determined in [6,12,13] and the rainbow connection number of graphs with fixed minimum degree was considered in [6,14,21,22]. Also, different other parameters similar to rainbow connection were introduced. Krivelevich and Yuster [14] introduced the concept of rainbow vertex-connection. Liu, Mestre and Sousa [20] proposed the concept of total rainbow connection. A vertex-coloured path is *vertex rainbow* if its internal vertices have distinct colours. A vertex-coloured graph G is *rainbow vertex-connected* if any two vertices are connected by a vertex rainbow path. In a totally-coloured graph G a path

* Corresponding author.
E-mail addresses: e.sidorowicz@wmie.uz.zgora.pl (E. Sidorowicz), eric.sopena@labri.fr (É. Sopena).

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is *total rainbow* if its edges and internal vertices have distinct colours. A totally-coloured graph G is *total rainbow connected* if any two vertices are connected by a total rainbow path. The *rainbow vertex-connection number* (resp. *total rainbow connection number*) of a connected graph G, denoted by $\mathrm{rvc}(G)$ (resp. $\mathrm{trc}(G)$), is the smallest number of colours needed to make G rainbow vertex-connected (resp. total rainbow connected).

The computational complexity of rainbow connectivity, rainbow vertex-connectivity and total rainbow connectivity was studied in [4,7,9,10,15]. It was shown that computing the rainbow connection number, the rainbow vertex-connection number and the total rainbow connection number of an arbitrary graph is NP-hard. Moreover, it was proved that it is already NP-complete to decide whether $\mathrm{rc}(G) = 2$, or $\mathrm{rvc}(G) = 2$, or $\mathrm{trc}(G) = 3$. See [18] or [19] for a survey about these different parameters.

The notions of rainbow connection and strong rainbow connection readily extend to digraphs, using arc-colouring instead of edge-colourings and directed paths (simply called paths in this paper) instead of paths. Note that in order to have bounded rainbow connection number, a digraph must be strongly connected.

Let D be a digraph. A *k-vertex-colouring*, a *t-arc-colouring* and a *p-total-colouring* of D is a mapping $c : V(D) \to \{1, \dots, k\}$, $\varphi : A(D) \to \{1, \dots, t\}$, $f : V(D) \cup A(D) \to \{1, \dots, p\}$, respectively. A *vertex-coloured* digraph, an *arc-coloured* digraph, a *totally-coloured* digraph, is then a pair (D, c), (D, φ), (D, f), respectively, where D is a digraph and c is a vertex-colouring, φ is an arc-colouring, and f is a total-colouring of D, respectively.

A path P in (D, φ) is *rainbow* if no two arcs of P are coloured with the same colour. If any two vertices in an arc-coloured digraph (D, φ) are connected by a rainbow path, then (D, φ) is said to be *rainbow connected* (or, equivalently, φ is a *rainbow arc-colouring* of G).

A path P in a vertex-coloured digraph (D, c) is *vertex rainbow* if its internal vertices have distinct colours. A vertex-coloured digraph (D, c) is *rainbow vertex-connected* (or, equivalently, c is a *rainbow vertex-colouring* of D) if any two vertices in (D, c) are connected by a rainbow path.

A path P in a totally-coloured digraph (D, f) is *total rainbow* if its edges and internal vertices have distinct colours. A totally-coloured digraph (D, f) is *total rainbow connected* (or, equivalently, f is a *rainbow total-colouring* of D) if any two vertices in (D, f) are connected by a rainbow path.

The *rainbow connection number* (*rainbow vertex-connection number*, *total rainbow connection number*) of a strong digraph D, is the minimum number of colours in a rainbow colouring (a rainbow vertex-colouring, a rainbow total-colouring). The rainbow connection number, the rainbow vertex-connection number, the total rainbow connection number are denoted by $\overrightarrow{\mathrm{rc}}(D)$, $\overrightarrow{\mathrm{rvc}}(D)$ and $\overrightarrow{\mathrm{trc}}(D)$, respectively.

The study of rainbow connection in oriented graphs (that is, antisymmetric digraphs) was initiated by Dorbec, Schiermeyer and the authors in [11] and then studied by Alva-Samos and Montellano-Ballesteros in [2,1,3]. Lei, Li, Liu and Shi [16] introduced the rainbow vertex-connection of digraphs. The total rainbow connection was studied in [17]. The strong version of rainbow connection was considered in [2,16,17,23].

In [11] it has been observed that the rainbow connection number of a digraph is upper bounded by its order and a characterization of all oriented graphs with rainbow connection number equal to their order has been given. However, this characterization does not lead to a polynomial time algorithm for the corresponding decision problem. In this paper we propose a new characterization of digraphs with rainbow connection number equal to their order. The obtained characterization shows that the problem of deciding whether a digraph of order n has rainbow connection number n can be solved in polynomial time. In contrast, Ananth, Nasre and Sarpatwar in [4] proved that the problem of deciding whether a digraph has rainbow connection number 2 is NP-complete.

Whereas it is easily observed that a graph G of order $n \geq 2$ has rainbow vertex-connection number at most $n - 2$, the rainbow vertex-connection number of a digraph can be equal to its order. We prove that a digraph D of order n has rainbow vertex-connection number n if and only if it has a Hamiltonian cycle C and three vertices with eccentricity $n - 1$ such that no two of them are consecutive vertices of C. For a digraph D of order n, the total rainbow connection number is at most $2n$ (see Section 3). We prove that a digraph D of order n has total rainbow connection number $2n$ if and only if it has rainbow vertex-connection number n.

This paper is organized as follows. After some preliminaries given in the next section, we give in Section 3 some basic results about the rainbow vertex-connection number and the total rainbow connection number of digraphs and some relations between these two parameters. In Section 4 we give some sufficient conditions for a digraph of order n to have rainbow connection number less than n, rainbow vertex-connection number less than n and total rainbow connection number less than $2n$. Finally, in Section 5, we propose a characterization of all digraphs of order n with rainbow connection number n, with rainbow vertex-connection number n, and with total rainbow connection $2n$, respectively.

2. Preliminaries

For a given digraph D, we denote by $V(D)$ and $A(D)$ its set of vertices and set of arcs, respectively. Given an arc xy in D, we say that y is an *out-neighbour* of x, while x is an *in-neighbour* of y. Moreover, we call x the *tail* of xy and y the *head* of xy. We denote by $N_D^+(x)$ the set of out-neighbours of x in D, and by $N_D^-(x)$ the set of in-neighbours of x in D. The *out-degree* $\deg_D^+(x)$ of x in D is the cardinality of its out-neighbourhood, that is $\deg_D^+(x) = |N_D^+(x)|$, and the *in-degree* $\deg_D^-(x)$ of x in D is the cardinality of its in-neighbourhood, that is $\deg_D^-(x) = |N_D^-(x)|$.

Given two digraphs D_1 and D_2, not necessarily vertex disjoint, we denote by $D_1 \cup D_2$ the digraph with vertex set $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$ and arc set $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$.

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