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Gallai's question and constructions of almost hypotraceable graphs

Gábor Wiener [a], Carol T. Zamfirescu [b,*]

[a] Department of Computer Science and Information Theory, Budapest University of Technology and Economics, Budapest, Hungary
[b] Department of Applied Mathematics, Computer Science and Statistics, Ghent University, Krijgslaan 281 - S9, 9000 Ghent, Belgium

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ABSTRACT

Consider a graph G in which the longest path has order $|V(G)| - 1$. We denote the number of vertices v in G such that $G - v$ is non-traceable with t_G. Gallai asked in 1966 whether, in a connected graph, the intersection of all longest paths is non-empty. Walther showed that, in general, this is not true. In a graph G in which the longest path has $|V(G)| - 1$ vertices, the answer to Gallai's question is positive iff $\mathrm{t}_G \neq 0$. In this article we study *almost hypotraceable* graphs, which constitute the extremal case $\mathrm{t}_G = 1$. We give structural properties of these graphs, establish construction methods for connectivities 1 through 4, show that there exists a cubic 3-connected such graph of order 28, and draw connections to works of Thomassen and Gargano et al.

1. Introduction

Throughout this paper all graphs are undirected, finite, connected, and contain neither loops nor multiple edges, unless explicitly stated otherwise. A graph G is *hypohamiltonian* (*hypotraceable*) if G does not contain a hamiltonian cycle (hamiltonian path) but for any vertex v in G the graph $G - v$ does contain a hamiltonian cycle (hamiltonian path). The study of hypohamiltonian graphs was initiated in the early sixties by Sousselier [22]. Many important results were obtained by Thomassen [24–28].

Kapoor, Kronk, and Lick [15] asked in 1968 whether hypotraceable graphs exist – in [17], Kronk stated that he "strongly feels" that they do not exist. This was refuted when a hypotraceable graph was subsequently found by Horton [13]. Thomassen [24,26] showed that there exists a hypotraceable graph with n vertices for $n \in \{34, 37\}$ and every $n \geq 39$, but we emphasise that Horton's graph has connectivity 3, whereas some of Thomassen's graphs have connectivity 2, others 3, depending on the construction method. (No 4-connected hypotraceable or hypohamiltonian graphs are known.) Since 1976, this list has been neither expanded – in particular, no hypotraceable graph of order smaller than 34 is known –, nor has it been shown to be complete.

Chvátal [5] asked whether *planar* hypohamiltonian graphs exist. Thomassen answered this question in the affirmative [26]. Based thereupon, he proved that planar hypotraceable graphs exist as well. In their survey on hypohamiltonian graphs, Holton and Sheehan [12] asked whether there is an order n' such that for every $n \geq n'$ there exists a planar hypohamiltonian graph on n vertices. We denote with n_0 the smallest such n'. Holton and Sheehan's question was settled by Araya and the first author [35] who showed that $n_0 \leq 76$. We now know that $23 \leq n_0 \leq 42$, see [10] and [14], respectively. Araya and the first author [35] also showed that there exists a planar hypotraceable graph on n vertices for every $n \geq 180$, which was improved to 156 in [14]. They also proved that there exists a planar hypotraceable graph on 162

* Corresponding author.
E-mail addresses: wiener@cs.bme.hu (G. Wiener), czamfirescu@gmail.com (C.T. Zamfirescu).

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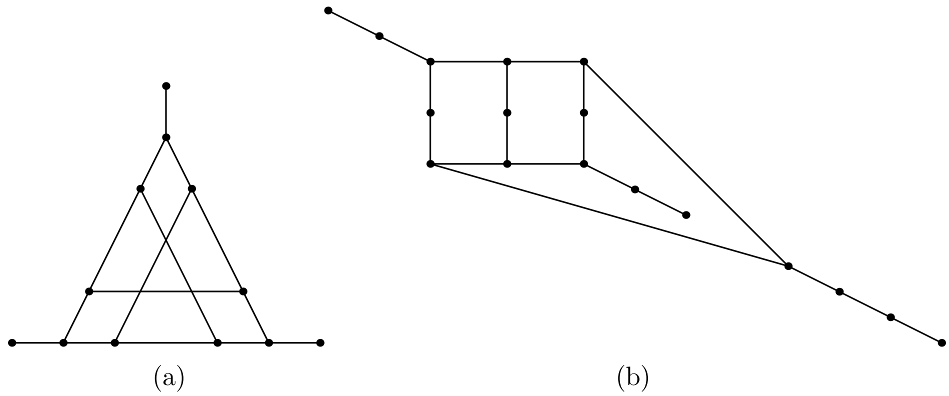

Fig. 1. (a) The Walther–Zamfirescu graph; (b) Schmitz' graph.

vertices, improving the previous bound of 186 from [38]. This was then lowered to 154 by Jooyandeh et al. [14]. Currently, the smallest planar hypotraceable graph (of order 138) is due to the first author [32,34], who used a new approach to construct hypotraceable graphs, explained at the end of the following paragraph. In a further article, Araya and the first author [1] showed that planar cubic hypotraceable graphs – in fact, these graphs are polyhedral, i.e., planar and 3-connected – on n vertices exist for $n = 340$ (this is the smallest example we know of) and every even $n \geq 356$, settling affirmatively an open question of Holton and Sheehan [12]. The number 356 was lowered to 344 by the second author [37].

For a graph G, we denote with $V(G)$ its vertex set and with $E(G)$ its edge set. In a graph G in which a longest cycle has length $|V(G)| - 1$, let $W \subset V(G)$ be the set of vertices w such that the graph $G - w$ is non-hamiltonian (and thus, for all $v \in V(G) \setminus W$, the graph $G - v$ is hamiltonian). We call $|W|$ the *hypohamiltonicity* of G, denote it by $\mathfrak{h}(G) = \mathfrak{h}_G$, and say that G is $\mathfrak{h}_G$-*hypohamiltonian*. A vertex from W is called *exceptional*. Until recently, all constructions of hypotraceable graphs relied on hypohamiltonian graphs as "building blocks". However, the first author showed [32,34] that 1-hypohamiltonian (also known as *almost hypohamiltonian*) graphs in which the exceptional vertex is cubic can be used as such building blocks, as well. Using this fact, he constructed the aforementioned smallest known planar hypotraceable graph.

As Kapoor, Kronk, and Lick [15], for a graph G we denote with $\partial(G)$ the length of a longest path in G. A graph G is *traceable* if it contains a hamiltonian path, i.e., $\partial(G) = |V(G)| - 1$, and for $v \in V(G)$, G is v-*traceable* if it contains a hamiltonian path with end-vertex v. In analogy to the definition given for cycles, consider a graph G with $\partial(G) = |V(G)| - 2$ and let $W \subset V(G)$ be the set of all vertices w such that the graph $G - w$ is non-traceable (and thus, for all $v \in V(G) \setminus W$, the graph $G - v$ is traceable). We call $|W|$ the *hypotraceability* $\mathfrak{t}(G) = \mathfrak{t}_G$ of G and say that G is $\mathfrak{t}_G$-*hypotraceable*.

It is easy to see that in any graph, two longest paths meet. Gallai [7] asked in 1966 whether *all* longest paths intersect. (Which is reminiscent of *Helly's property*: a collection of sets satisfies it, if any sub-collection of pairwise intersecting sets has a nonempty intersection.) We follow Chen et al. [4], and call a vertex present in all longest paths of a given graph a *Gallai vertex*, and the set of all Gallai vertices the *Gallai set*.

It turns out that, in general, the answer to Gallai's question is negative. It was Walther [29] who first showed that there exists a graph in which the intersection of all longest paths is empty, i.e., a graph with empty Gallai set. A few years later, a significantly smaller example – of order 12 – was independently found by Walther and T. Zamfirescu, see [11,30,39]. It is shown in Fig. 1(a). Brinkmann and Van Cleemput [3] proved (using computers) that there is no smaller example.

The smallest known example of a *planar* graph in which all longest paths have empty intersection has order 17 and is due to Schmitz [20], see Fig. 1(b). More variants of Gallai's problem were discussed, for instance by asking arbitrary *pairs* of vertices to be missed, by demanding higher connectivity, or by posing the question for graphs which can be embedded in various lattices. For a survey, see [21].

Let us emphasise the connection between hypotraceability and Gallai vertices: consider a graph G with $\partial(G) = |V(G)| - 2$. This is extremal in the sense that it is the greatest length of a longest path for which Gallai's question is interesting. Then the hypotraceability of G is precisely the cardinality of the Gallai set of G. Hence, in the (extremal) family of graphs G with $\partial(G) = |V(G)| - 2$, the answer to Gallai's question is positive if and only if $\mathfrak{t}_G \neq 0$.

Gallai's question has drawn much attention; we give here only a small selection of results. For an overview, see [21]. One central direction of research was – since with Walther's result, in general, Gallai's question has a negative answer – to study in which families of graphs Gallai's question had a *positive* answer. We call such graphs, ad hoc, *good*. Trees, for instance, abide. Klavžar and Petkovšek [16] proved that if every block of G is *hamiltonian-connected*, i.e., any two vertices are the end-vertices of a hamiltonian path, then G is good. Balister, Győri, Lehel, and Schelp [2] showed that circular arc graphs – a graph G is a *circular arc graph* if there exists a mapping α of $V(G)$ into a collection of arcs of a circle such that, for every $v, w \in V(G)$, there is an edge between v and w if and only if $\alpha(v) \cap \alpha(w) \neq \emptyset$ – are good, as well. (Note that this includes interval graphs. According to Rautenbach and Sereni [19], there is a gap in the proof of Balister et al. For details, see [19].)