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Approximating weighted induced matchings

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ABSTRACT

An induced matching is a matching where no two edges are connected by a third edge. Finding a maximum induced matching on graphs with maximum degree Δ, for $\Delta \geq 3$, is known to be NP-complete. In this work we consider the weighted version of this problem, which has not been extensively studied in the literature. We devise an almost tight fractional local ratio algorithm with approximation ratio Δ, which proves to be effective also in practice. Furthermore, we show that a simple greedy algorithm applied to $K_{1,k}$-free graphs yields an approximation ratio $2k-3$. We explore the behavior of this algorithm on subclasses of chair-free graphs and we show that it yields an approximation ratio k when restricted to $(K_{1,k}$, chair)-free graphs.

1. Introduction

Let $G = (V, E)$ be a simple undirected graph. A subset $\mathcal{M} \subseteq E$ is an *induced matching* if $\mathcal{M}$ is a matching and no edge of E connects two edges of $\mathcal{M}$. The Maximum Induced Matching (MIM) problem is to find an induced matching of maximum cardinality. Note that the size of a MIM is the same as the size of a maximum independent set of $L(G)^2$, the square of the line graph of G. To our best knowledge, MIM was introduced in [30] among some generalizations of the classical maximum matching problem, although it was referred to as "risk-free" marriage problem. The authors proved it to be NP-complete when restricted to bipartite graphs of degree at most 4. Independently, this problem was introduced in [3], where a polynomial time algorithm for chordal graphs was given. Thereafter MIM was polynomially solved on trees [12], circular-arc graphs [14], trapezoid, k-interval dimension, cocomparability graphs [15], ($\mathrm{Star}_{1,2,3}$, Sun_4)-free bipartite graphs [26], weakly chordal graphs [5] and on bounded treewidth graphs [27]. The relations between the families of G and $L(G)^2$ were exploited to conclude that MIM is polynomially solvable for polygon–circle, AT-free, and filament-interval graphs; where the latter contains cocomparability, circle, circular-arc, chordal and outerplanar graphs [4]. A polynomial time algorithm for hhd-free graphs and a linear time algorithm for a subclass of hhd-free graphs which is a more general class than chordal graphs were given in [24]. The problem was proven to be NP-complete on bipartite graphs of maximum degree 3, C_4-free bipartite graphs [26], d-regular graphs for $d \geq 5$, line graphs (which implies that MIM is NP-complete on claw-free graphs and chair-free graphs) [23] and on cubic planar graphs [11,22]. Note that since any claw-free graph is $K_{1,k}$-free for $k \geq 3$ and cannot contain a chair, it follows that MIM is NP-complete on $(K_{1,k}$, chair)-free graphs — a class that we address in this work. Several algorithms were given for classes related to AT-free graphs, in particular it was shown that the Maximum Weight Induced Matching (MWIM) problem is polynomially solvable on graphs with bounded asteroidal index [8]. It was observed that orthogonal ray graphs have bounded asteroidal index [31], which by [8] implies a polynomial time algorithm for MWIM for this class. MIM was solved in polynomial time for line graphs of Hamiltonian graphs, and it

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was shown that the problem remains NP-complete on Hamiltonian graphs [23]. In the same paper the authors gave some polynomial time algorithms for subclasses of P_5-free graphs and they noted that if G is P_5-free, then $L(G)^2$ is P_5-free. In those years was still an open question whether the maximum independent set on P_5-free graphs was polynomial time solvable, and therefore this observation did not directly yield a polynomial time algorithm for MIM on P_5-free graphs. Remarkably, this long standing question was affirmatively answered a decade later in [25]; which implies that MIM is polynomial on P_5-free graphs. In [23] a polynomial time algorithm was given for recognizing graphs where the size of a maximum induced matching is the same as the size of a maximum matching, and a polynomial time algorithm was given to find a MIM in such class. A simpler recognition algorithm for this class was given in [6]. The line of research of [6,23] was continued in [10], where the authors simplified further the proofs of [6] and gave a polynomial time algorithm for recognizing graphs where the maximum induced matching differs in at most k with the maximum matching. MWIM was shown to be polynomially solvable on chordal graphs, (claw, net)-free graphs and some other subclasses of claw-free graphs [2].

Regarding approximability, for graphs of degree at most Δ only a $2(\Delta-1)$-approximation is known [32]. However, several approximations were given for d-regular graphs with $d \geq 3$, which all used the same theoretical upper bound for proving their approximation ratios — namely, that any induced matching of a d-regular graph has at most $m/(2d-1)$ edges. This bound was first introduced in [32], where the author gave a simple greedy algorithm with performance ratio $d - \frac{1}{2} + \frac{1}{4d-2}$. This was improved in [11], where the authors gave an asymptotic approximation ratio $d-1$. In the same work they gave a PTAS for planar graphs of degree at most 3. In the following year, a $0.75d + 0.15$ approximation ratio for d-regular graphs was given [16]. This ratio was further improved to $0.708\overline{3}d + 0.425$ on (C_3, C_5)-free d-regular graphs [29]. An algorithm for cubic graphs with performance ratio $9/5$ appears in [20].

Several results were given regarding lower bounds and algorithms attaining them. Any subcubic planar graph has an induced matching of size at least $m/9$ and it is possible to find such induced matching [21]. It is known that in subcubic graphs without short cycles there must be an induced matching of size at least $(n-1)/5$ [17]. For bounded degree graphs there is a polynomial time algorithm that computes an induced matching of size at least $\frac{n}{(\lceil\Delta/2\rceil+1)(\lfloor\Delta/2\rfloor+1)}$ for graphs with sufficiently large Δ and with no isolated vertices [18]. One can find an induced matching in polynomial time with at least $m/20$ edges for graphs with degree at most 4, and at least $m/18$ edges for a subclass of these graphs [19].

On the negative side, MIM cannot be approximated on general graphs with a constant performance ratio unless P = NP [32]. Furthermore, the problem cannot be approximated within a factor $n^{1/2-\epsilon}$ for some $\epsilon > 0$, unless P = NP [28]. Moreover, MIM is APX-complete for d-regular bipartite graphs for $d \geq 3$ [9].

Despite the vast amount of research done for induced matchings, not much has been achieved for the weighted version of this problem besides [2,8,31]. To our best knowledge, no approximation algorithm was given using a linear program as an upper bound. However, a generalization of MWIM with edge capacities was considered in [13], and for some particular cases (that excluded the classical induced matching) the authors gave some constant approximation ratios by relating a natural linear programming formulation with a capacitated b-matching polytope.

In this work we propose a fractional local ratio algorithm for MWIM with performance ratio Δ. For an overview on local ratio algorithms we suggest the survey [1]. Furthermore, we show that a simple greedy algorithm yields an approximation ratio $2k-3$ for $K_{1,k}$-free graphs and k for $(K_{1,k}$, chair)-free graphs.

2. A Δ-approximation algorithm

Let $G = (V, E)$ be an edge-weighted graph with weights $w_e \in \mathbb{Q}_{\geq 0}$. For an edge $uv = e \in E$ we define $N(e) = N(v) \cup N(u)$ and $\delta(e) = \delta(u) \cup \delta(v)$, where $N(v)$ is the open neighborhood of v and $\delta(v)$ is the set of edges incident to v. We denote $C(e) \subseteq E$ to be the set of edges which are *in conflict* with e; more formally, $C(e) = \bigcup_{w \in N(e)} \delta(w)$. Note that in our definition $e \in C(e)$.

We can model the MWIM problem with an integer linear program. We define the binary variables x_e for $e \in E$ such that e is included in the solution if and only if $x_e = 1$. Consider the following formulation, where $x(A)$ denotes $\sum_{a \in A} x_a$.

$$\begin{array}{lrcll} \max & \sum_{e\in E} w_e x_e & & & \\ \text{s.t.} & x(\delta(e)) & \leq & 1 & \forall e \in E, \\ & x_e & \in & \{0,1\} & \forall e \in E. \end{array}$$

Our algorithm uses linear relaxations of the above program restricted to different sets of variables. Formally, for a subset $F \subseteq E$, we denote LP_F to be the following linear program restricted to the variables x_f for $f \in F$.

$$\begin{array}{lrcll} \max & \sum_{e\in E} w_e x_e & & & \\ \text{s.t.} & x(\delta(e)) & \leq & 1 & \forall e \in E, \\ & x_e & \geq & 0 & \forall e \in F. \end{array}$$

In what follows we show that Algorithm 1 has performance ratio Δ provided that in each recursive call one can find an edge $e \in F$ such that $x(C(e)) \leq \Delta$. Note that there is at most one recursive call in each call, and in each step $|F|$ decreases by at least one. Therefore, the algorithm ends after at most $|E|$ recursive calls. For the sake of completeness, we include the proof of the following theorem.