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Bounds on the 2-domination number

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ABSTRACT

In a graph G, a set $D \subseteq V(G)$ is called 2-dominating set if each vertex not in D has at least two neighbors in D. The 2-domination number $\gamma_2(G)$ is the minimum cardinality of such a set D. We give a method for the construction of 2-dominating sets, which also yields upper bounds on the 2-domination number in terms of the number of vertices, if the minimum degree $\delta(G)$ is fixed. These improve the best earlier bounds for any $6 \leq \delta(G) \leq 21$. In particular, we prove that $\gamma_2(G)$ is strictly smaller than $n/2$, if $\delta(G) \geq 6$. Our proof technique uses a weight-assignment to the vertices where the weights are changed during the procedure.

1. Introduction

We study the graph invariant $\gamma_2(G)$, called 2-domination number, which is in close connection with the fault-tolerance of networks. Our main contributions are upper bounds on $\gamma_2(G)$ in terms of the number of vertices, when the minimum degree $\delta(G)$ is fixed. The earlier upper bounds of this type are tight for $\delta(G) \leq 4$, here we establish improvements for the range of $6 \leq \delta(G) \leq 21$. Our approach is based on a weight-assignment to the vertices, where the weights are changed according to some rules during a 2-domination procedure.

1.1. Basic terminology

Given a simple undirected graph G, we denote by $V(G)$ and $E(G)$ the set of its vertices and edges, respectively. The *open neighborhood* of a vertex $v \in V(G)$ is defined as $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, while the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. Then, the *degree* $d(v)$ is equal to $|N(v)|$ and the *minimum degree* of G is the smallest vertex degree $\delta(G) = \min\{d(v) \mid v \in V(G)\}$. We say that a vertex v *dominates* itself and its neighbors, that is exactly the vertices contained in $N[v]$. A set $D \subseteq V(G)$ is a *dominating set* if each vertex of G is dominated or equivalently, if the closed neighborhood of D, defined as $N[D] = \bigcup_{v \in D} N[v]$, equals $V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of such a set D. Domination theory has a rich literature, for results and references see the monograph [13].

There are two different natural ways to generalize the notion of (1-)domination to multiple domination. As defined in [10], a *k-dominating set* is a set $D \subseteq V(G)$ such that every vertex not in D has at least k neighbors in D. Moreover, D is a *k-tuple dominating set* if the same condition $|N[v] \cap D| \geq k$ holds not only for all $v \in V(G) \setminus D$ but for all $v \in V(G)$. The minimum cardinalities of such sets are the *k-domination number* $\gamma_k(G)$ and the *k-tuple domination number* of G, respectively.

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1.2. 2-domination and applications

A sensor network can be modeled as a graph such that the vertices represent the sensors and two vertices are adjacent if and only if the corresponding devices can communicate with each other. Then, a dominating set D of this graph G can be interpreted as a collection of cluster-heads, as each sensor which does not belong to D has at least one head within communication distance.

A k-dominating set D may represent a dominating set which is $(k-1)$-fault tolerant. That is, in case of the failure of at most $(k-1)$ cluster-heads, each remaining vertex is either a head or keeps in connection with at least one head. The price of this k-fault tolerance might be very high. In the extremal case, when k is greater than the maximum degree in the network, the only k-dominating set is the entire vertex set. But for the usual cases arising in practice, 2-domination might be enough and it does not require extremely many heads.

Note that k-tuple domination might need much more vertices (cluster-heads) than k-domination. As proved in [11], for each real number $\alpha > 1$ and each natural number n large enough, there exists a graph G on n vertices such that its k-tuple domination number is at least $\frac{k}{\alpha}$ times larger than its k-domination number. There surely exist some practical problems where k-tuple domination is needed, but for many problems arising k-domination seems to be sufficient. Indeed, if a cluster-head fails and is deleted from the network, we may not need further heads to supervise it. This motivates our work on the 2-domination number γ_2.

Another potential application of our results in sensor networks concerns the data collection problem. Here, each sensor has two capabilities: either measures and reports, or receives and collects data. Only one position from those two can be active at the same time. After deploying, the organization process determines exactly which sensors supply the measuring and the collector function in the given network. Since it is a natural condition that every measurement should be saved in at least two different devices, the set of collector sensors should form a 2-dominating set in the network.

We mention shortly that many further kinds of application exist. For example a facility location problem may require that each region is either served by its own facility or has at least two neighboring regions with such a service [17]. In this context, facility location may also mean allocation of a camera system, or that of ambulance service centers.

1.3. Upper bounds on the 2-domination number

Although this subject attracts much attention (see the recent survey [8] for results and references) and it seems very natural to give upper bounds for γ_2 in terms of the minimum degree, there are not too many results of this type. The following general upper bounds are known. (As usual, n denotes the order of the graph, that is the number of its vertices.)

- If the minimum degree $\delta(G)$ is 0 or 1, then $\gamma_2(G)$ can be equal to n.
- If $\delta(G) = 2$, then $\gamma_2(G) \le \frac{2}{3}\, n$. This follows from a general upper bound on $\gamma_k(G)$ proved in [9]. The bound is tight for graphs each component of which is a K_3.
- If $\delta(G) \ge 3$, then $\gamma_2(G) \le \frac{1}{2}\, n$. The general theorem, from which the bound follows, was established in [7]. Note that a 2-dominating set of cardinality at most $n/2$ can be constructed by a simple algorithm. We divide the vertex set into two parts and then in each step, a vertex which has more neighbors in its own part than in the other one, is moved into the other part. If the minimum degree is at least 3, this procedure results in two disjoint 2-dominating sets. Note that for $\delta(G) = 3$ and 4 the bound is tight. For example, it is easy to check that $\gamma_2(K_4) = 2$ and $\gamma_2(K_4 \Box K_2) = 4$.[1]
- For every graph G of minimum degree $\delta \ge 0$,

$$\gamma_2(G) \le \frac{2\ln(\delta+1)+1}{\delta+1}\, n.$$

  This upper bound was obtained in [12] using probabilistic method and it is a strong result when δ is really high. On the other hand, it gives an upper bound better than $0.5\, n$ only if $\delta(G) \ge 11$.

In this paper we present a method which can be used to improve the existing upper bounds when the minimum degree is in the "middle" range. Particularly, we show that if $\delta(G) \ge 6$ then $\gamma_2(G)$ is strictly smaller than $n/2$; $\delta(G) = 7$ implies $\gamma_2(G) < 0.467\, n$; $\delta(G) = 8$ implies $\gamma_2(G) < 0.441\, n$; and $\gamma_2(G) < 0.418\, n$ holds for every graph whose minimum degree is at least 9.

The paper is organized as follows. In Section 2, we state our main theorem and its corollaries which are the new upper bounds for specified minimum degrees. In Section 3 our main theorem is proved. Finally, we make some remarks on the algorithmic aspects of our results.

2. Our results

To avoid the repetition of the analogous argumentations for different minimum degrees, we will state our theorem in a general form which is quite technical. Then, the upper bounds will follow as easy consequences. First, we introduce a set of

[1] The Cartesian product $K_4 \Box K_2$ is the graph of order 8 which consists of two copies of K_4 with a matching between them. Note that $\gamma_3(K_4 \Box K_2)$ also equals 4.