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Old and new results on packing arborescences in directed hypergraphs

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ABSTRACT

We propose a further development in the theory of packing arborescences. First we review some of the existing results on packing arborescences and then we provide common generalizations of them to directed hypergraphs. We introduce and solve the problem of reachability-based packing of matroid-rooted hyperarborescences and we also solve the minimum cost version of this problem. Furthermore, we introduce and solve the problem of matroid-based packing of matroid-rooted mixed hyperarborescences.

1. Introduction

We study packings of arborescences in this paper. An ***r*-arborescence** is a directed tree on a vertex-set containing the **root** vertex r in which each vertex has in-degree 1 except r. Throughout this paper, by packing subgraphs in a directed (hyper)graph, we mean a set of arc-disjoint subgraphs. (For other definitions, see the next section.) The starting point of the research on arborescence-packings is the following famous result of Edmonds [6] on packing spanning arborescences.

Theorem 1 ([6]). *There exists a packing of k spanning r-arborescences in a digraph $\vec{G} = (V, A)$ if and only if*

$$\varrho_A(X) \geq k \tag{1}$$

holds for all $\emptyset \neq X \subseteq V \setminus r$ where $\varrho_A(X)$ denotes the in-degree of X. ■

This result has extensions in many directions. For our purposes let us mention four of them: the result of Kamiyama, Katoh, Takizawa [14] on packing *reachability* arborescences (Theorem 4 in this paper), Theorem 5 on packing matroid-rooted arborescences with *matroid* constraint by Durand de Gevigney, Nguyen, Szigeti [5], Theorem 3 on packing spanning *hyper* arborescences (Frank, T. Király, Z. Király [10]) and Theorem 7 on packing spanning *mixed* arborescences (Frank [8]). Fig. 1 shows all possible combinations of these extensions. The results without citations corresponding to black boxes of the diagram are presented in this paper, the ones in gray are yet to be proved to be in P (see Section 7.1).

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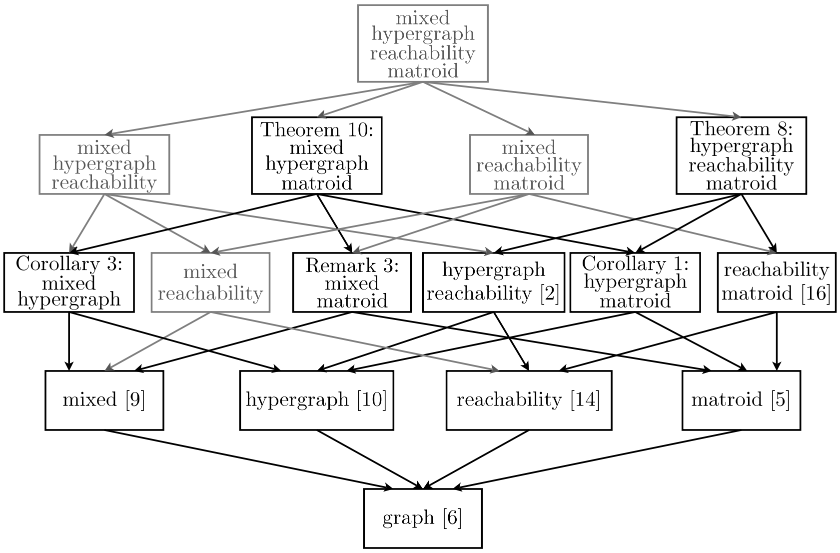

Fig. 1. All possible common generalizations of the 4 problems mentioned in the introduction.

The main contribution of this work is to show how the existing hypergraphical results can be derived directly from their graphical counterparts. We note that the original proofs of these results were different. Both Frank, Z. Király and T. Király [10] and Bérczi and Frank [1] showed that a directed hypergraph satisfying their condition for the packing problem can be reduced – by an operation called *trimming* – to a digraph satisfying the condition of the graphical counterpart of their problem.

Our method looks a bit similar to this; however, we also add some extra vertices to the digraph to ensure that the condition of the graphical result holds automatically for the digraph if the hypergraphical condition holds for the directed hypergraph. We also note that this method allows us to find a minimum cost solution of these problems for any cost function on the set of directed hyperedges.

Using the same method, we solve the problem of reachability-based packing of matroid-rooted hyperarborescences, that is, a common generalization of three of the above four extensions, excluding the mixed one. We also consider a generalization of other three of the above four extensions, excluding the reachability one this time, namely the problem of matroid-based packing of matroid-rooted mixed hyperarborescences. Using a new orientation result (Theorem 11) on hypergraphs covering intersecting supermodular functions, we reduce this problem to its directed version, the problem of matroid-based packing of matroid-rooted hyperarborescences, which in turn is a special case of the problem of reachability-based packing of matroid-rooted hyperarborescences.

2. Definitions

In this paper, $\boldsymbol{\mathcal{H}} = (V, \mathcal{E})$ will be a hypergraph. We assume that all the hyperedges in $\mathcal{E}$ are of size at least 2. When all the hyperedges are of size 2, that is, when the hypergraph is a graph, we will denote it by $\boldsymbol{G} = (V, E)$. For a vertex set X, $\boldsymbol{i_{\mathcal{E}}(X)}$ denotes the number of hyperedges in $\mathcal{E}$ that are contained in X. For a partition $\boldsymbol{\mathcal{P}} = \{V_0, V_1, \dots, V_\ell\}$ of V, where only $\boldsymbol{V_0}$ can be empty, we denote by $\boldsymbol{e_{\mathcal{E}}(\mathcal{P})}$ the number of hyperedges in $\mathcal{E}$ intersecting at least two members of $\boldsymbol{\mathcal{P}}$.

Let $\vec{\boldsymbol{\mathcal{H}}} = (V, \mathcal{A})$ be a directed hypergraph (**dypergraph** for short) where $\boldsymbol{V}$ denotes the set of vertices and $\boldsymbol{\mathcal{A}}$ denotes the set of dyperedges of $\vec{\mathcal{H}}$. By a **dyperedge** we mean a pair (Z, z) such that $z \in Z \subseteq V$, where z is the **head** of the dyperedge (Z, z) and the elements of $Z \setminus z$ are the **tail**s of the dyperedge (Z, z). We assume that each dyperedge has one head and at least one tail. When a dypergraph is a digraph, we will denote it by $\vec{\boldsymbol{G}} = (V, A)$. Let $X \subseteq V$. We say that the dyperedge (Z, z) **enters** X if the head of (Z, z) is in X and at least one tail of (Z, z) is not in X. We define the **in-degree** $\boldsymbol{\varrho_{\mathcal{A}}(X)}$ of X as the number of dyperedges in $\mathcal{A}$ entering X.

For a set function h on V, we say that the dypergraph $\vec{\mathcal{H}}$ **covers** h if

$$\varrho_{\mathcal{A}}(X) \geq h(X) \text{ for all } X \subseteq V. \tag{2}$$

By **trimming** the dypergraph $\vec{\mathcal{H}}$ we mean replacing each dyperedge (Z, z) of $\vec{\mathcal{H}}$ by an arc tz where t is one of the tails of the dyperedge (Z, z).

By an **orientation** of $\mathcal{H}$, we mean a dypergraph $\vec{\mathcal{H}}$ obtained from $\mathcal{H}$ by choosing, for every $Z \in \mathcal{E}$, an orientation of Z, that is by choosing a head z for Z.

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