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# Spanning tree with lower bound on the degrees

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ABSTRACT

We concentrate on some recent results of Egawa and Ozeki (2015, 2014), and He et al. (2002). We give shorter proofs and polynomial time algorithms as well.

We present two new proofs for the sufficient condition for having a spanning tree with prescribed lower bounds on the degrees, achieved recently by Egawa and Ozeki (2015). The first one is a natural proof using induction, and the second one is a simple reduction to the theorem of Lovász (1970). Using an algorithm of Frank (1975) we show that the condition of the theorem can be checked in time $O(m\sqrt{n})$, and moreover, in the same running time – if the condition is satisfied – we can also generate the spanning tree required. This gives the first polynomial time algorithm for this problem.

Next we show a nice application of this theorem for the simplest case of the Weak Nine Dragon Tree Conjecture, and for the game coloring number of planar graphs, first discovered by He et al. (2002).

Finally, we give a shorter proof and a polynomial time algorithm for a good characterization of having a spanning tree with prescribed degree lower bounds, for the special case when $G[S]$ is a cograph, where $S$ is the set of the vertices having degree lower bound prescription at least two. This theorem was proved by Egawa and Ozeki in 2014 while they did not give a polynomial time algorithm.



## 1. Introduction

Let $G = (V, E)$ be a simple undirected graph, $S \subseteq V$ and $f : S \to \{2, 3, 4, \ldots\}$ be an integer-valued function on $S$. For a subset $X$ of vertices let $f(X) = \sum_{x\in X} f(x)$. For disjoint sets of vertices $X$ and $Y$, $d_G(X, Y)$ denotes the number of edges between $X$ and $Y$, $d_G(X) = d_G(X, V - X)$ and $d_G(u) = d_G(\{u\})$. When the graph $G$ is clear from the context, we omit it from the notation.

The open neighborhood is denoted by $\Gamma_G(X) = \{u \in V - X \mid \exists x \in X,\ ux \in E\}$, and the closed neighborhood is denoted by $\Gamma^*_G(X) = \Gamma_G(X) \cup X$. A subgraph induced by a vertex set $X \subseteq V$ is denoted by $G[X]$, the number of its edges by $i_G(X)$, and the number of its components by $c(G[X])$ or $c_G(X)$. We will use the convention that $\Gamma_G(\emptyset) = \emptyset$ and $c(G[\emptyset]) = 0$.

Egawa and Ozeki proved the following sufficient condition for having a forest (or spanning tree) with prescribed lower bounds on the degrees.

**Theorem 1** ([2]). *If for all nonempty subsets $X \subseteq S$ we have $|\Gamma^*_G(X)| > f(X)$ then there is a forest subgraph $F$ of $G$, such that for all vertices $v \in S$ we have $d_F(v) \geq f(v)$.*

**Corollary 2** ([2]). *If for all nonempty subsets $X \subseteq S$ we have $|\Gamma^*_G(X)| > f(X)$ and $G$ is connected, then there is a spanning tree $T$ of $G$, such that for all vertices $v \in S$ we have $d_T(v) \geq f(v)$.*

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Special cases of this theorem appeared in the literature as follows. When $G$ is bipartite and $S$ is one of the classes, it was proved by Lovász in 1970 [9]. For general $G$, if $S$ is a stable set, it was proved by Frank in 1976 [4] in a stronger form, as in this case the condition above is also necessary, giving a special case of Theorem 9 in Section 6. For a not necessary stable $S$ a stronger condition is proved to be sufficient by Singh and Lau [11], namely: $|\Gamma^*_G(X)| > f(X) + c_G(X)$.

Deciding whether for a triplet $(G, S, f)$ there is a spanning tree $T$ with degree lower bounds, i.e., $d_T(v) \geq f(v)$ for all $v \in S$, is NP-complete (let $S = V - \{u, v\}$ and $f(x) = 2$ for each $x \in S$; any appropriate spanning tree is a Hamiltonian path). However, consider the following algorithmic problem. For given $(G, S, f)$ check, whether the condition of Corollary 2 is satisfied, and if yes, then construct the appropriate spanning tree $T$. We show that this problem is polynomially solvable, namely in time $O(m\sqrt{n})$, where $n = |V|$ and $m = |E|$.

In the next section we give a simpler proof than that of Egawa and Ozeki, using induction. In Section 3 we give another proof, that is a simple reduction to the theorem of Lovász, yielding also a fast algorithm, detailed in Section 4. In Section 5 we show an application (as an example) of Theorem 1 for the game coloring number of some planar graphs. Finally, in Section 6 we show how we can use these ideas to prove a good characterization of Egawa and Ozeki [1] for a special case. Our proof is not only shorter but also yields the first polynomial time algorithm for this case.

## 2. First proof—by induction

We prove Theorem 1 by induction on the number of edges. If $G$ is a forest or $S = \emptyset$ then the theorem is obviously true.

We call a set $X \subseteq S$ tight if it satisfies the condition $|\Gamma^*_G(X)| \geq f(X) + 1$ with equality.

If $uv$ is an edge and $G - uv$ satisfies the condition then we are done by induction. So we may assume that for every edge $uv$ the graph $G - uv$ has a set $X \subseteq S$ violating the condition (a violating set). This implies that there are no edges outside $S$, and also that for each edge $uv$ either $u$ or $v$ is contained in a tight set $X$, where the other one is connected to $X$ by exactly one edge. If $u$ is contained in tight set $X$ with $d(v, X) = 1$ then we orient edge $uv$ from $v$ to $u$, otherwise, from $u$ to $v$. (If both $u$ and $v$ are contained in such a tight set, we choose arbitrarily.) This oriented graph $\vec{G}$ has the property that no arc leaves $S$. (The word *arc* will always refer to a directed edge, in this section a directed edge of $\vec{G}$.) The in-degree of a vertex $u$ (set $X$) is denoted by $\varrho(u)$ (or $\varrho(X)$ resp.).

**Claim 3.** *For each $u \in S$ we have $f(u) \geq \varrho(u)$.*

**Proof of Claim.** If $\varrho(u) > 0$ then $u$ is contained in a tight set. As $|\Gamma^*_G(X)|$ is a submodular set function, the intersection and the union of two intersecting tight sets are both tight. Thus the intersection $I(u)$ of all tight sets containing $u$ is also a tight set. Every arc $vu$ of $\vec{G}$ was oriented this way because it entered a tight set containing $u$, consequently, it must enter $I(u)$ as well.

If $|I(u)| = 1$ then, by the tightness, we have $f(u) = d_G(u) \geq \varrho(u)$. Otherwise, as $I(u) - u$ is not a violating set, if $vu$ is an arc of $\vec{G}$, then the vertex $v$ does not have any neighbors in $I(u) - u$. Thus we have $f(I(u)) + 1 - f(u) = f(I(u) - u) + 1 \leq |\Gamma^*_G(I(u) - u)| \leq |\Gamma^*_G(I(u))| - \varrho(u) = f(I(u)) + 1 - \varrho(u)$, giving the claim. □

To finish the proof of the theorem it is enough to prove that $G$ is a forest. Suppose this is not the case. Choose a cycle $C$ which minimizes $|V(C) - S|$. Let $X = V(C) \cap S$ and let $\overline{X}$ be the closure of $X$ relative to $S$: $\overline{X} = \{v \in S \mid \exists x \in X$, such that $v$ and $x$ are in the same component of $G[S]\}$. Clearly $c(G[\overline{X}]) \leq c_G(X)$ and, by the observation made above, no arc leaves $\overline{X}$.

If $V(C) \subseteq S$ then, using Claim 3 and the fact that $G[\overline{X}]$ is now connected and contains a cycle, $f(\overline{X}) \geq i_G(\overline{X}) + \varrho(\overline{X}) \geq |\overline{X}| + \varrho(\overline{X}) \geq |\Gamma^*_G(\overline{X})|$, and this contradicts to the assumption of the theorem.

Otherwise, $G[S]$ is a forest and $|V(C) - S| \geq c(G[\overline{X}])$. Now, by Claim 3, $f(\overline{X}) \geq i_G(\overline{X}) + \varrho(\overline{X}) \geq |\overline{X}| - c(G[\overline{X}]) + \varrho(\overline{X})$. As $V - S$ is an independent set and no arc leaves $S$, at least two arcs go from any vertex of $V(C) - S$ to $\overline{X}$, that is $\overline{X}$ has at most $\varrho(\overline{X}) - |V(C) - S| \leq \varrho(\overline{X}) - c(G[\overline{X}])$ different neighbors in $V - S$. Thus $|\Gamma^*_G(\overline{X})| \leq |\overline{X}| + \varrho(\overline{X}) - c(G[\overline{X}]) \leq f(\overline{X})$, a contradiction again. □ □

## 3. Second proof—reduction to Lovász' theorem

In this section we prove Theorem 1 using a theorem of Lovász [9]. We quote this old theorem reformulated for fitting the notation used in this paper. We denote by $f^+$ the function $f + 1$, i.e., $f^+(x') = f(x') + 1$ for $x' \in S'$.

**Theorem 4** ([9]). *Let $B = (S' \cup V, E')$ be a bipartite graph and $f : S' \to \{2, 3, 4, \ldots\}$ be a function. $B$ has a forest subgraph $F_0$ with the property $d_{F_0}(x') = f^+(x')$ for every $x' \in S'$, if and only if for all nonempty $X' \subseteq S'$ we have $|\Gamma_B(X')| > f(X')$.*

**Proof of Theorem 1.** We have $(G, S, f)$ given, and let $S'$ be a set disjoint from $V$ with elements $S' = \{u' \mid u \in S\}$, and extend $f$ to $S'$ in the obvious way: $f(u') := f(u)$ for each $u \in S$.

Construct a bipartite graph $B = (V \cup S', E')$ as follows. For each ordered vertex pair $(u \in S,\ v \in V)$ we put an edge $u'v$ into $E'$ if $uv \in E$, and we also put the 'vertical' edges $u'u$ for each $u \in S$. Observe that for each $X \subseteq S$ (if $X'$ denotes the corresponding subset of $S'$) we have $\Gamma_B(X') = \Gamma^*_G(X)$. Therefore the condition of Theorem 4 is satisfied and thus we have a forest subgraph $F_0$ of $B$ with $d_{F_0}(x') = f^+(x')$ for every $x' \in S'$.

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