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Lexicographical polytopes

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ABSTRACT

Within a fixed integer box of $\mathbb{R}^n$, lexicographical polytopes are the convex hulls of the integer points that are lexicographically between two given integer points. We provide their descriptions by means of linear inequalities.

Throughout, ℓ, u, r, s will denote integer points satisfying $\ell \leq r \leq u$ and $\ell \leq s \leq u$, that is r and s are within $[\ell, u]$. A point $x \in \mathbb{Z}^n$ is *lexicographically smaller than* $y \in \mathbb{Z}^n$, denoted by $x \preccurlyeq y$, if $x = y$ or the first nonzero coordinate of $y - x$ is positive. We write $x \prec y$ if $x \preccurlyeq y$ and $x \neq y$. The *lexicographical polytope* $P_{\ell,u}^{r\preccurlyeq s}$ is the convex hull of the integer points within $[\ell, u]$ that are lexicographically between r and s:

$$P_{\ell,u}^{r\preccurlyeq s} = conv\{x \in \mathbb{Z}^n : \ell \leq x \leq u, r \preccurlyeq x \preccurlyeq s\}.$$

The *top-lexicographical polytope* $P_{\ell,u}^{\preccurlyeq s} = conv\{x \in \mathbb{Z}^n : \ell \leq x \leq u, x \preccurlyeq s\}$ is the special case when $r = \ell$. Similarly, the *bottom-lexicographical polytope* is $P_{\ell,u}^{r\preccurlyeq} = conv\{x \in \mathbb{Z}^n : \ell \leq x \leq u, r \preccurlyeq x\}$.

Given $a, u \in \mathbb{R}_+^n$ and $b \in \mathbb{R}_+$, the *knapsack polytope* defined by $K_u^{a,b} = conv\{x \in \mathbb{Z}^n : \mathbf{0} \leq x \leq u, ax \leq b\}$ is *superdecreasing* if:

$$\sum_{i>k} a_i u_i \leq a_k \quad \text{for } k = 1, \ldots, n. \tag{1}$$

Close relations between top-lexicographical and superdecreasing knapsack polytopes appear in the literature. For the 0/1 case, that is when $\ell = \mathbf{0}$ and $u = \mathbf{1}$, Gillmann and Kaibel [2] first noticed that top-lexicographical polytopes are special cases of superdecreasing knapsack ones, and the converse has been later established by Muldoon et al. [5]. Recently, Gupte [3] generalized the latter result by showing that all superdecreasing knapsacks are top-lexicographical polytopes.

To prove this last statement, Gupte [3] observes that a superdecreasing knapsack $K_u^{a,b}$ is the top-lexicographical polytope $P_{\mathbf{0},u}^{\preccurlyeq s}$, where s the lexicographically greatest integer point of $K_u^{a,b}$. The non trivial inclusion actually holds because every integer point x of $P_{\mathbf{0},u}^{\preccurlyeq s}$ satisfies $ax \leq as$. Indeed, by definition, if $x \prec s$, there exists $k \in \{1, \ldots, n\}$ such that $x_k + 1 \leq s_k$ and $x_i = s_i$

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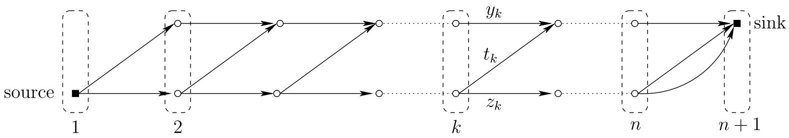

Fig. 1. Path representation of the points of $X^{\preccurlyeq s}_{\ell,u}$.

for $i < k$. Hence, we have $b - ax \geq as - ax \geq \sum_{i>k} a_i(s_i - x_i) + a_k \geq \sum_{i>k} a_i(s_i - x_i + u_i) \geq 0$, because of (1), $s_i \geq 0$ and $u_i \geq x_i$.

It turns out that top-lexicographical polytopes are superdecreasing knapsack polytopes. Indeed, let $P^{\preccurlyeq s}_{\ell,u}$ be a top-lexicographical polytope for some s within $[\ell, u]$. Possibly after translating, we may assume $\ell = \mathbf{0}$. Define a by $a_k = \sum_{i>k} a_i u_i + 1$, for $k = 1, \ldots, n$, and let $b = as$. Since the associated knapsack polytope $K^{a,b}_u$ is superdecreasing, if $x \preccurlyeq s$ then $ax \leq as = b$, for all x within $[\mathbf{0}, u]$. Moreover, the converse holds because, inequalities (1) being all strict, $s \prec x$ implies $b = as < ax$. Therefore, $P^{\preccurlyeq s}_{\mathbf{0},u} = K^{a,b}_u$. These observations are summarized in the following.

Observation 1. *Superdecreasing knapsacks are top-lexicographical polytopes, and conversely (up to translations).*

Motivated by a wide range of applications, such as knapsack cryptosystems [6] or binary expansion of bounded integer variables (*e.g.,* [8, p. 477]), several papers are devoted to the polyhedral description of these families of polytopes. For the 0/1 case, the description appeared in [4] from the knapsack point of view. It was later rediscovered from the lexicographical point of view in [2,5]. Moreover, Muldoon et al. [5] and Angulo et al. [1] independently showed that intersecting a 0/1 top- with a 0/1 bottom-lexicographical polytope yields the description of the corresponding lexicographical polytope. Recently, these results were generalized for the bounded case by Gupte [3].

In this paper, we provide the description of the lexicographical polytopes using extended formulations. Our approach provides alternative proofs of the aforementioned results of Gupte [3].

The outline of the paper is as follows. In Section 1, we provide a flow based extended formulation of the convex hull of the componentwise maximal points of a top-lexicographical polytope. Projecting this formulation is surprisingly straightforward, and thus we get the description in the original space. In Section 2, using the fact that a top-lexicographical polytope is, up to translation, the submissive of the above convex hull, we derive the description of top-lexicographical polytopes. We then show that a lexicographical polytope is the intersection of its top- and bottom-lexicographical polytopes.

1. Convex hull of componentwise maximal points

From now on, $X^{\preccurlyeq s}_{\ell,u}$ will denote the set of the points $p^i = (s_1, \ldots, s_{i-1}, s_i - 1, u_{i+1}, \ldots, u_n)$, for $i = 1, \ldots, n+1$ such that $s_i > \ell_i$, where $p^{n+1} = s$ by definition. Note that $X^{\preccurlyeq s}_{\ell,u}$ consists of the componentwise maximal integer points of $P^{\preccurlyeq s}_{\ell,u}$, to which we added, for later convenience, the point $p^n = (s_1, \ldots, s_{n-1}, s_n - 1)$ if $s_n > \ell_n$.

1.1. A flow model for $X^{\preccurlyeq s}_{\ell,u}$

We first model the points of $X^{\preccurlyeq s}_{\ell,u}$ as paths from 1 to $n+1$ in the digraph given in Fig. 1.

Our digraph is composed of $n+1$ layers, each containing two nodes except the first and the last ones. There are three arcs connecting the layer k to the layer $k+1$, an upper arc y_k, a diagonal arc t_k and a lower arc z_k. The only exception concerns the first level, which does not have the upper arc.

The arcs connecting two successive layers correspond to a coordinate of $x \in X^{\preccurlyeq s}_{\ell,u}$. More precisely, given a directed path P from 1 to $n+1$, we define the point x by setting, for $k = 1, \ldots, n$,

$$x_k = \begin{cases} u_k & \text{if } y_k \in P, \\ s_k - 1 & \text{if } t_k \in P, \\ s_k & \text{if } z_k \in P. \end{cases}$$

As shown in Observation 2, the set of (x, y, z, t) satisfying the following set of inequalities is an extended formulation of $conv(X^{\preccurlyeq s}_{\ell,u})$:

$$x_i = u_i y_i + (s_i - 1)t_i + s_i z_i \qquad \text{for } i = 1, \ldots, n, \tag{2}$$

$$y_1 = 0 \tag{3}$$

$$y_i = y_{i-1} + t_{i-1} \qquad \text{for } i = 2, \ldots, n, \tag{4}$$

$$z_i = z_{i+1} + t_{i+1} \qquad \text{for } i = 1, \ldots, n-1, \tag{5}$$

$$t_i = 0 \qquad \text{whenever } s_i = \ell_i, \tag{6}$$

$$y_n + t_n + z_n = 1 \tag{7}$$

$$y_i, t_i, z_i \geq 0 \qquad \text{for } i = 1, \ldots, n. \tag{8}$$

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