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The hierarchy of circuit diameters and transportation polytopes

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ABSTRACT

The study of the diameter of the graph of polyhedra is a classical problem in the theory of linear programming. While transportation polytopes are at the core of operations research and statistics it is still unknown whether the Hirsch conjecture is true for general $m\times n$-transportation polytopes. In earlier work the first three authors introduced a hierarchy of variations to the notion of graph diameter in polyhedra. This hierarchy provides some interesting lower bounds for the usual graph diameter.

This paper has three contributions: First, we compare the hierarchy of diameters for the $m\times n$-transportation polytopes. We show that the Hirsch conjecture bound of $m+n-1$ is actually valid in most of these diameter notions. Second, we prove that for $3\times n$-transportation polytopes the Hirsch conjecture holds in the classical graph diameter. Third, we show for $2\times n$-transportation polytopes that the stronger monotone Hirsch conjecture holds and improve earlier bounds on the graph diameter.

1. Introduction

Transportation problems are among the oldest and most fundamental problems in mathematical programming, operations research, and statistics [10–13,17,18]. An $m\times n$-transportation problem has m supply points and n demand points to be met. Each supply point holds a quantity $u_i > 0$ and each demand point needs a quantity $v_j > 0$. If $y_{ij} \geq 0$ describes the flow from the supply point i to the demand point j, then the set of feasible flow assignments, $y \in \mathbb{R}^{m\times n}$, can be described by

$$\begin{aligned} \sum_{j=1}^{n} y_{ij} &= u_i \quad i=1,\ldots,m, \\ \sum_{i=1}^{m} y_{ij} &= v_j \quad j=1,\ldots,n, \\ y_{ij} &\geq 0 \quad i=1,\ldots,m,\ j=1,\ldots,n. \end{aligned}$$

The set of solutions to these constraints constitutes a *transportation polytope*. Here the vectors u and v are called the *marginals* or *margins* for the transportation polytope, and a point y inside the polytope is called a *feasible flow assignment*.

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The standard transportation problem requires the optimization of a linear objective function over this set. A common way to solve these problems is the simplex algorithm [9]. In the context of a worst-case performance of the simplex method, the study of the *graph diameter* (or combinatorial diameter) of polyhedra is a classical field in the theory of linear programming. This is the diameter of its underlying *1-skeleton*. Hence the *graph distance* between two vertices (0-faces) in P is the minimum number of edges (1-faces) needed to go from one vertex to the other, and the graph diameter of P is the maximum distance between its vertices. The connection to the simplex algorithm becomes even more direct when investigating the "monotone diameter". Here the diameter is realized by a monotone path on the same graph for a given linear functional. This monotone path is an edge-walk visiting vertices whose objective function values are non-decreasing with respect to the functional.

In 1957, W. Hirsch stated his famous conjecture (see e.g., [9]) claiming that the diameter of a polytope is at most $f-d$, where d is its dimension and f its number of facets. In his recent celebrated work, Santos finally gave a counterexample for general polytopes [21], but the Hirsch conjecture is true for some classes of polytopes. A survey is found in [16]. In particular, it holds for dual transportation polyhedra [1] and for 0, 1-polytopes [20]. For the latter, the even stronger monotone Hirsch conjecture (or monotonic bounded Hirsch conjecture), asking whether $f-d$ is a bound on the monotone diameter, was shown to be true [19]. But it is still open whether the Hirsch conjecture holds for $m\times n$-transportation polytopes despite a long line of research which we outline next.

For $m\times n$-transportation polytopes, the Hirsch conjecture states an upper bound of $m+n-1$. This bound is valid for $m = 2$ [10] and for some special classes of transportation polytopes [2], even with respect to the monotone diameter for the so-called signature polytopes [4]. Further, a much lower bound holds for the partition polytopes, a special class of 0, 1-transportation polytopes [5]. This is a generalization of the well-known fact that the assignment polytope has diameter 2 for $n \geq 4$ [3]. For $m \geq 3$, the best known bounds for $m\times n$-transportation polytopes are in fact linear: a bound of $8(m+n-2)$ is presented in [8], and this is improved to $4(m+n-1)$ (it remains unpublished but a sketch of the proof is shown in [10]).

In an attempt to understand the behavior of the graph diameter we introduced a hierarchy of distances and diameters for polyhedra that extend the usual edge walk [6,7]: Instead of only going along actual edges of the polyhedron, we walk along *circuits*, which are all potential edge directions of the polyhedron. This means in particular that we can possibly enter the interior of the polyhedron.

The transportation polytopes are of the form $P = \{z \in \mathbb{R}^d : Az = b, z \geq 0\}$ for an integral matrix A. Then the *circuits* or *elementary vectors* associated with A are those vectors $g \in \ker(A) \setminus \{0\}$, for which g is support-minimal in the set $\ker(A) \setminus \{0\}$, where g is normalized to (coprime) integer components. These are exactly all the edge directions of $\{z \in \mathbb{R}^n : Az = b, z \geq 0\}$ that appear when letting b vary. For vertices v, w of P, we call a sequence $v = y^{(0)}, \ldots, y^{(k)} = w$ a *circuit walk of length k*, if for all $i = 0, \ldots, k-1$ we have $y^{(i+1)} - y^{(i)} = \alpha_i g^i$ for some circuit g^i and some $\alpha_i > 0$. The *circuit distance* from v to w is the minimum length of a circuit walk from v to w. The *circuit diameter* of P is the maximum circuit distance between any two vertices of P. In the following we prove lower and upper bounds on the circuit diameters of transportation polytopes using various notions of diameters. We here look at different notions of circuit diameter that arise by putting further restrictions on the circuit walks as introduced in [7] and further discussed in [22]; we consider four main types of circuit distance:

- $\mathcal{CD}$ (the circuit walks do not have to satisfy any additional properties),
- $\mathcal{CD}_f$ (all points in the circuit walk have to be feasible points in the polyhedron),
- $\mathcal{CD}_{fm}$ (maximal feasible steps are applied, that is, for all i, $y^{(i)} + \alpha_i g^i \in P$, but $y^{(i)} + \alpha g^i \notin P$ for all $\alpha > \alpha_i$; this is the circuit distance introduced in [6]),
- $\mathcal{CD}_e$ (the circuit steps go along the edges from vertex to vertex; this distance corresponds to the graph diameter and was denoted $\mathcal{CD}_{efm}$ in [7]).

These four ways to measure the distance between vertices form the 'central chain' in the larger hierarchy of distances shown in [7]. Note that they satisfy the relation

$$\mathcal{CD}_e \geq \mathcal{CD}_{fm} \geq \mathcal{CD}_f \geq \mathcal{CD} \ ,$$

by the simple fact that the definitions become less restrictive.

We are interested in the diameters of transportation polytopes with respect to these notions of distances. The subscript we use in $\mathcal{CD}_*$ changes according with the distance being employed. For simplicity of notation, we use $\mathcal{CD}_*$ with slightly different meanings, which are clear in the context of the presentation. It can mean

- the actual distance between two specific vertices within a polytope,
- the diameter of a specific transportation polytope,
- or the maximal diameter of any polytope in a given family of transportation polytopes.

Hereby, the above hierarchy directly translates to a hierarchy of diameters for a given polytope, as well as a hierarchy of maximal diameters in a family of transportation polytopes. Our first results in this paper are lower and upper bounds on the 'bottom' part of this hierarchy of diameters. For $\mathcal{CD}_e$ and $\mathcal{CD}_{fm}$ it is open whether there is a polynomial bound in the bit size of their inequality description for general polytopes, whereas for $\mathcal{CD}_f$ and $\mathcal{CD}$ the Hirsch conjecture bound holds for all polytopes [7]. We refine this statement for transportation polytopes as follows.